ASBG-Colourings of Unicyclic and Cycle-Edge-Disjoint Graphs

Cian O'Brien
Rachel Quinlan and Kevin Jennings

Postgraduate Modelling Research Group
National University of Ireland, Galway

c.obrien40@nuigalway.ie

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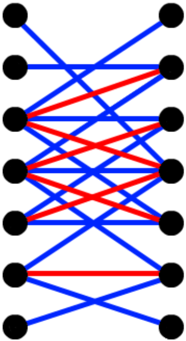
An ordering of the vertices is allowable if the vertices of each part can be embedded in that order on two parallel lines in the plane such that the edges incident with each vertex alternate in colour (beginning and ending with blue) in that embedding.

Alternating Signed Bipartite Graphs



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Some necessary (but not sufficient) conditions for a given graph G to have an ASBG-colouring c are:

- G must be bipartite and balanced;
- Each vertex of G must have odd degree. This is because each vertex of G^c must have blue degree one higher than red degree.

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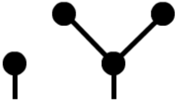

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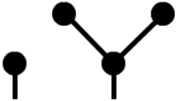

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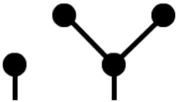

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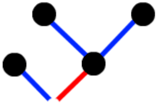

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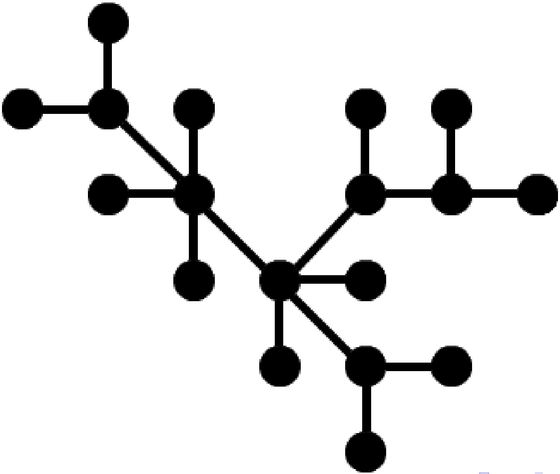
Theorem: A tree T is ASBG-colourable iff leaf-twig configurations can be removed until *the trivial ASBG remains*.



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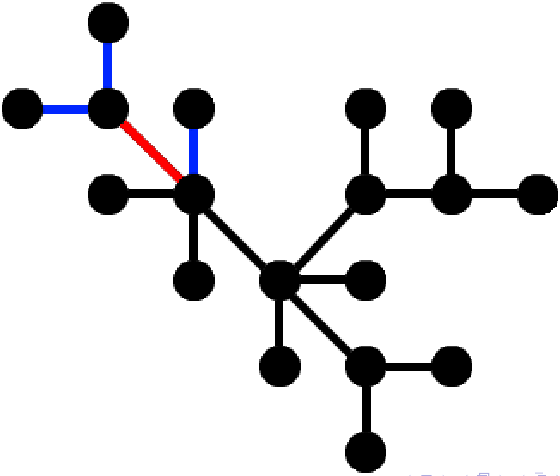

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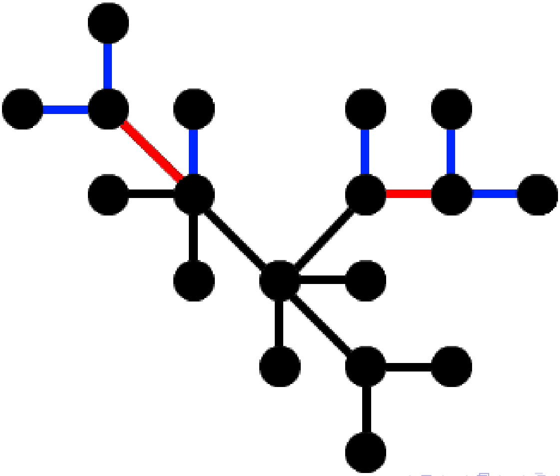

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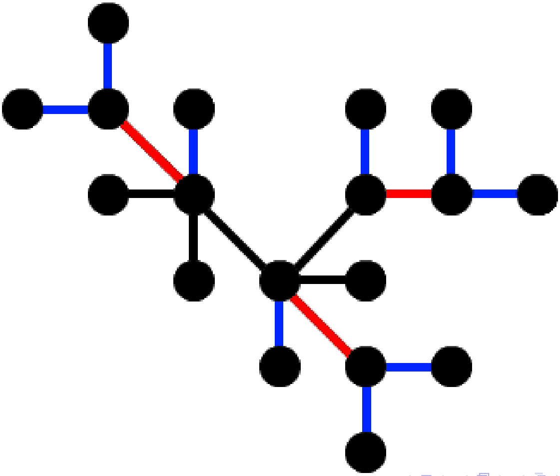

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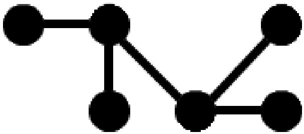

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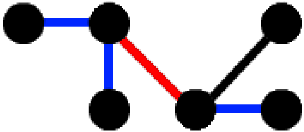

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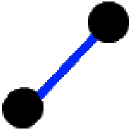

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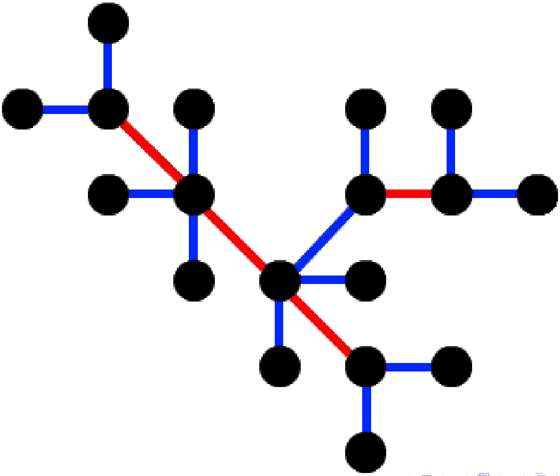

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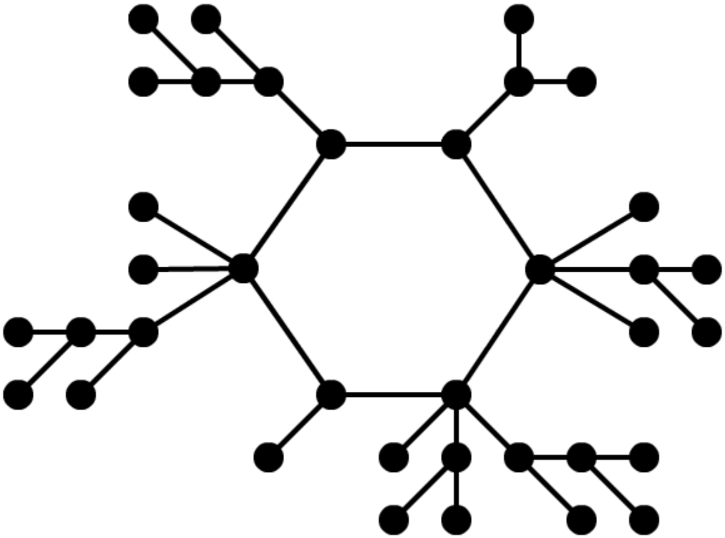

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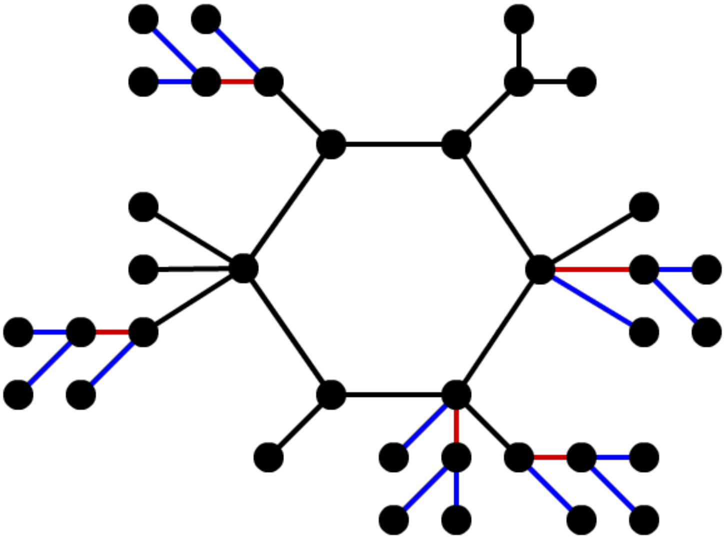

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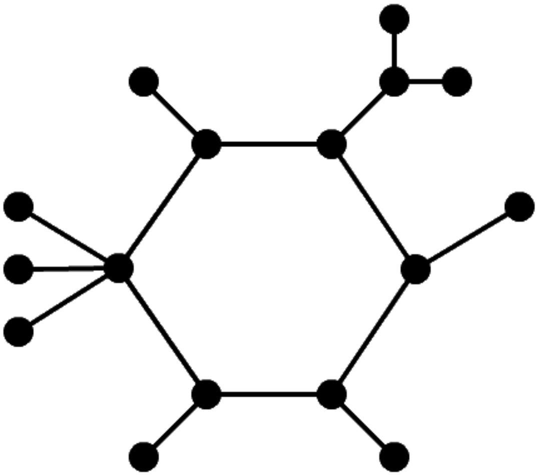

Leaf, Twig, and Triple-Type Vertices

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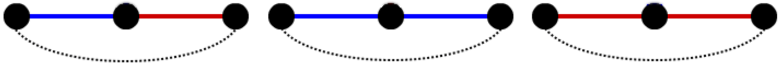

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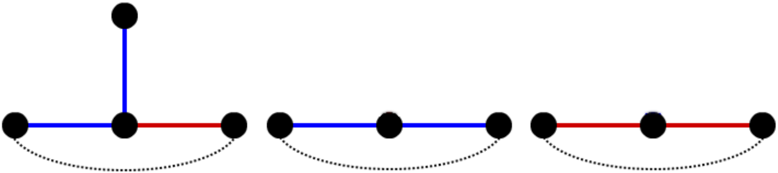

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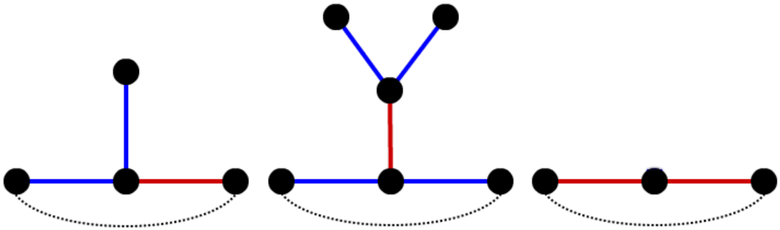

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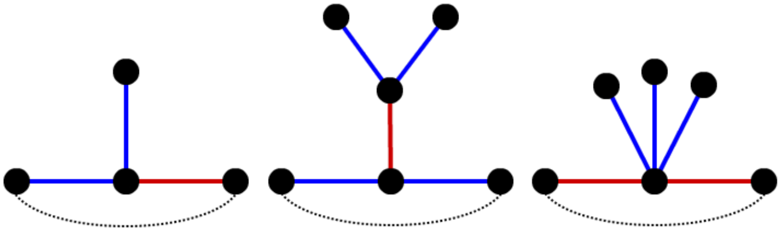

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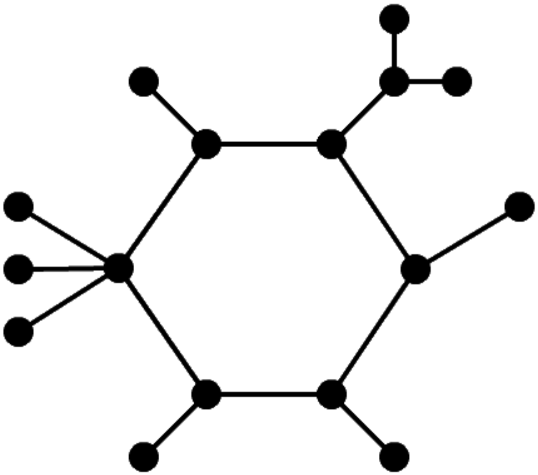

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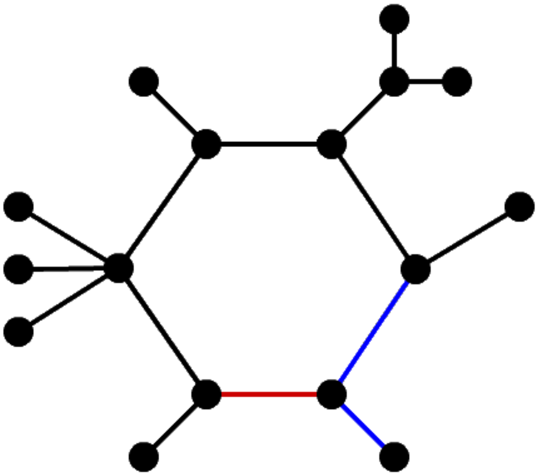

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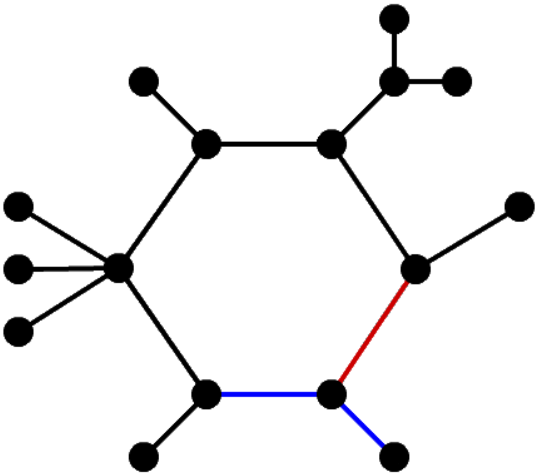

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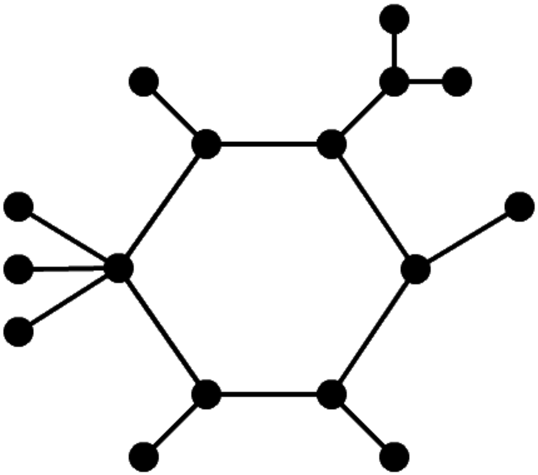

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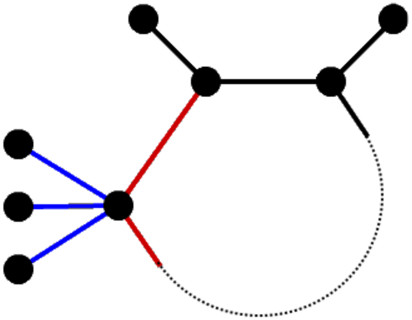

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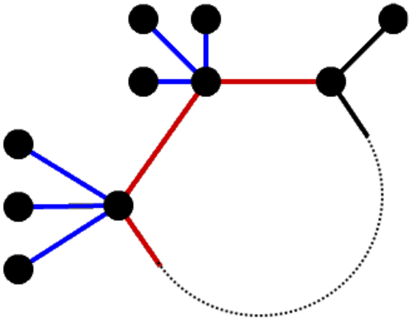

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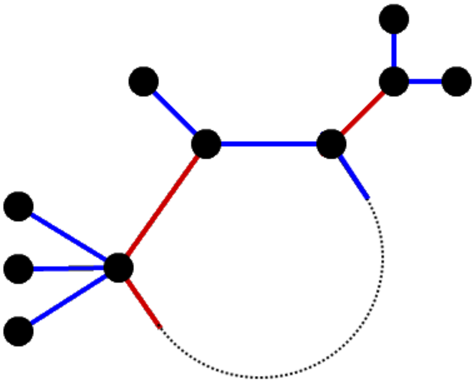

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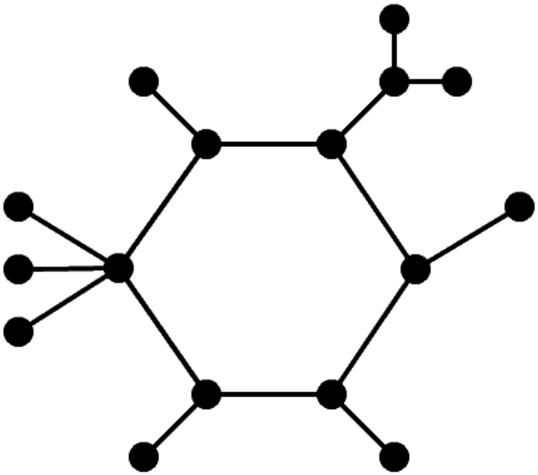

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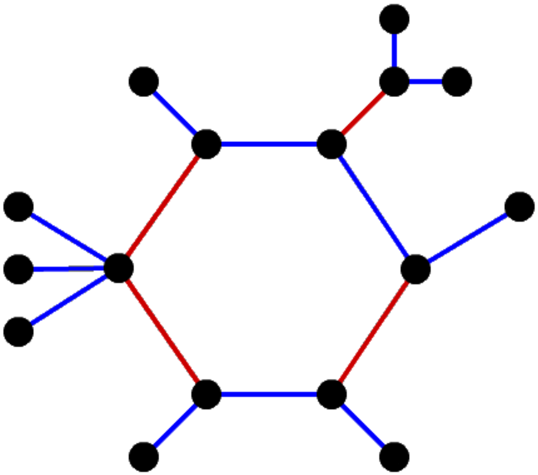

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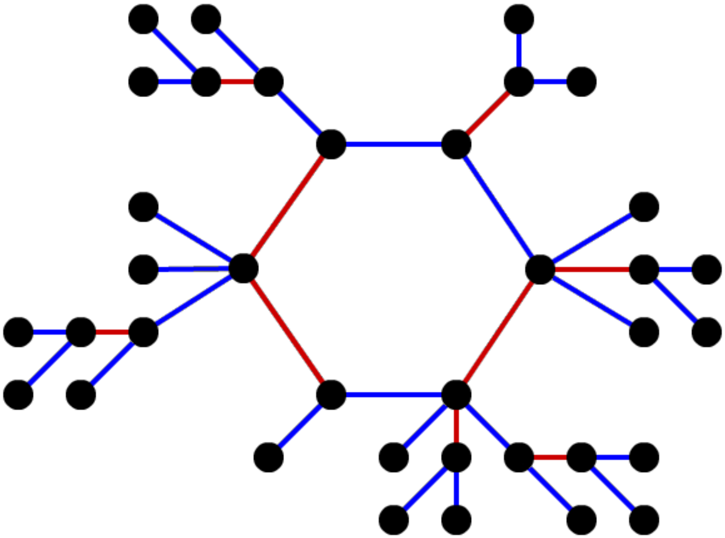

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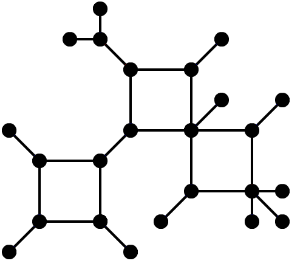

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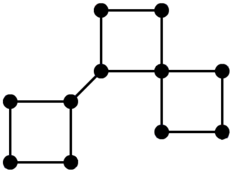

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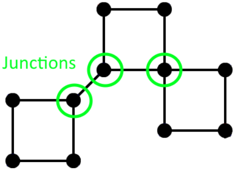

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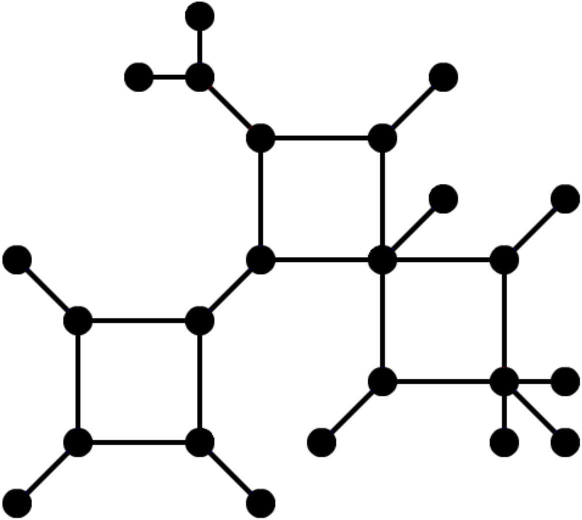
As well as leaf, twig, and triple-type vertices, cycle-edge-disjoint graphs (and all graphs in general) have one other type of vertex: a junction.

Junction Weight Algorithm



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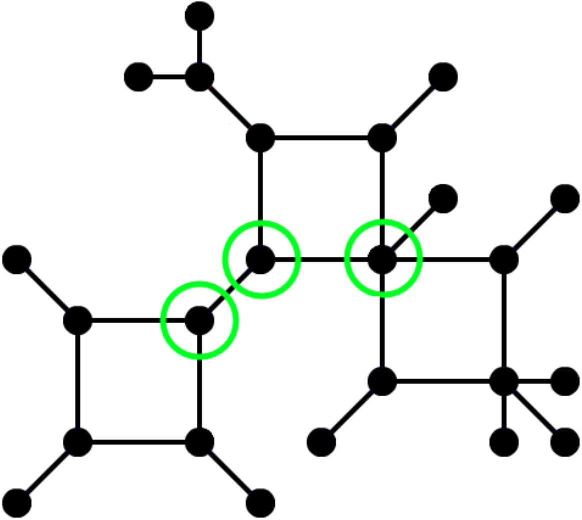

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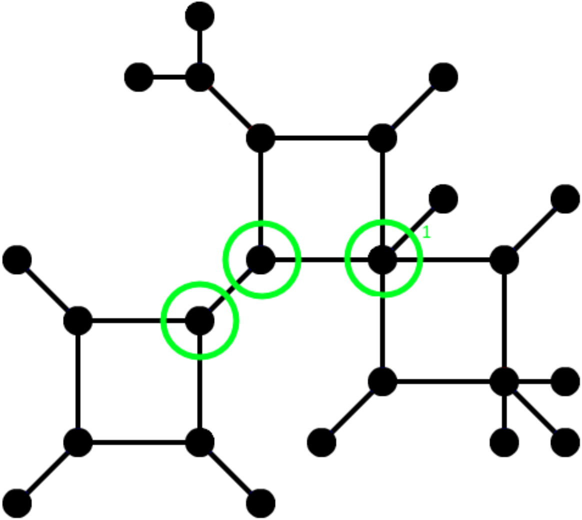

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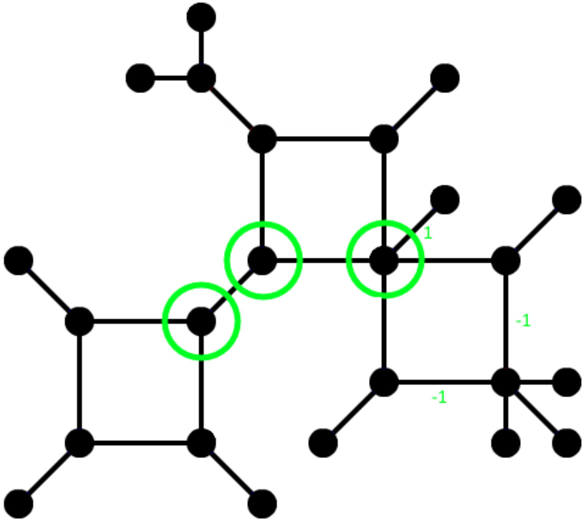

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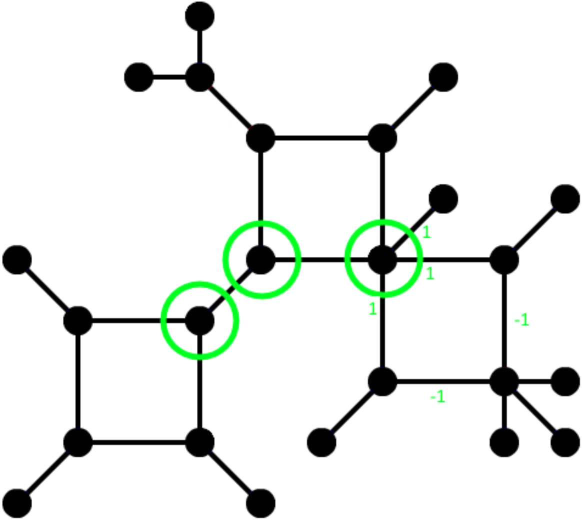

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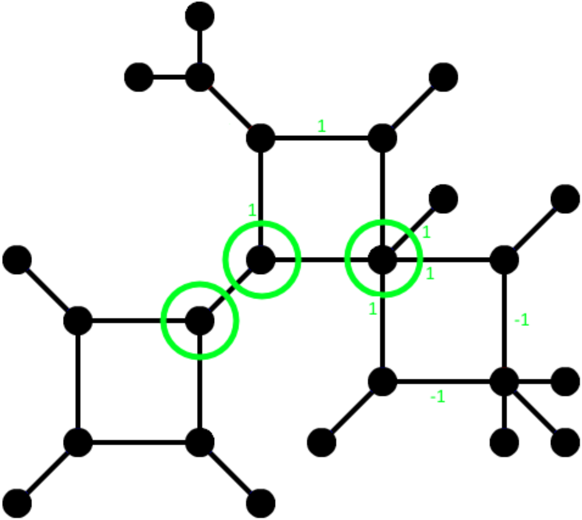

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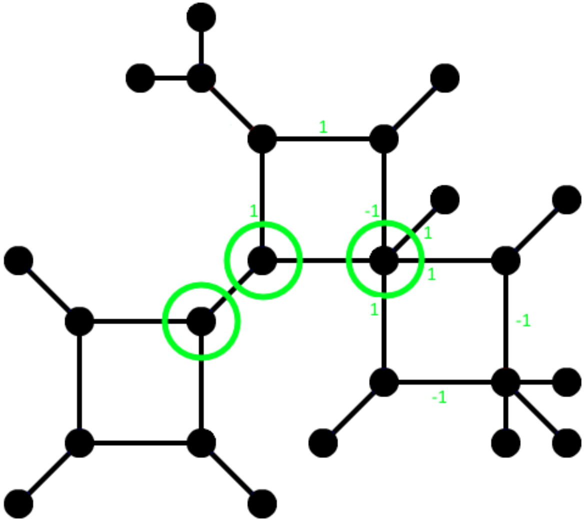

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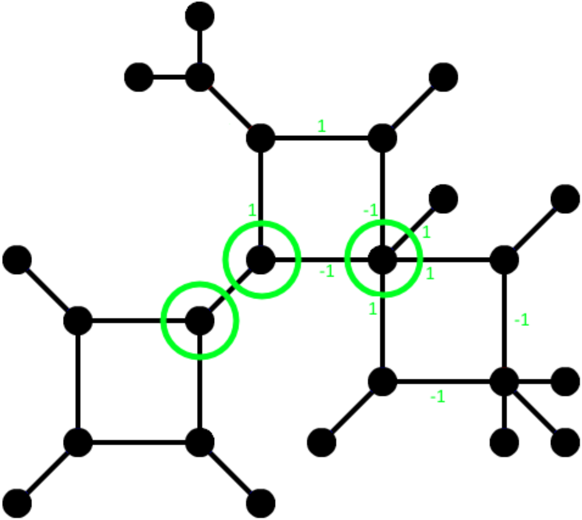

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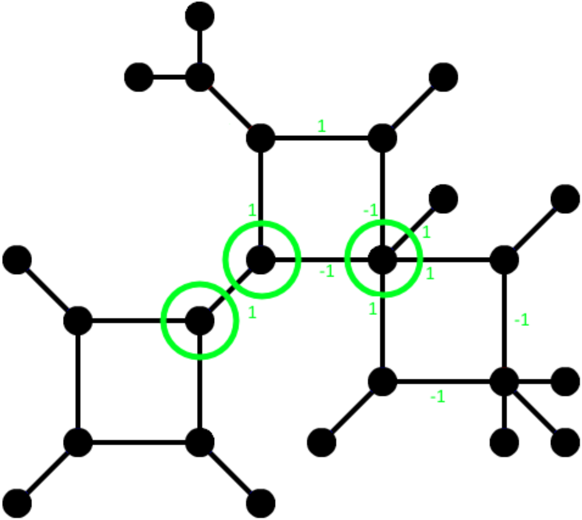

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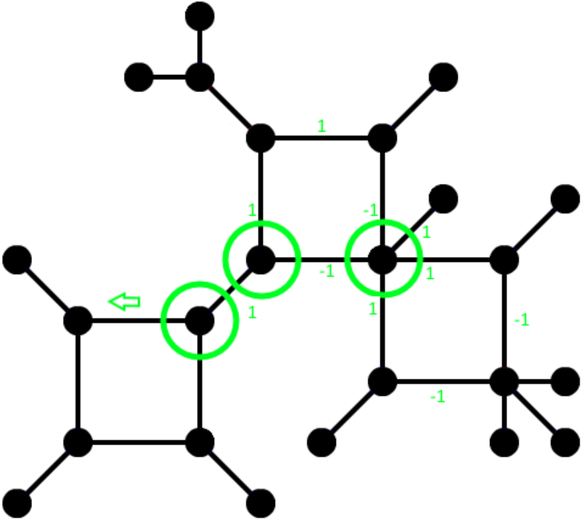

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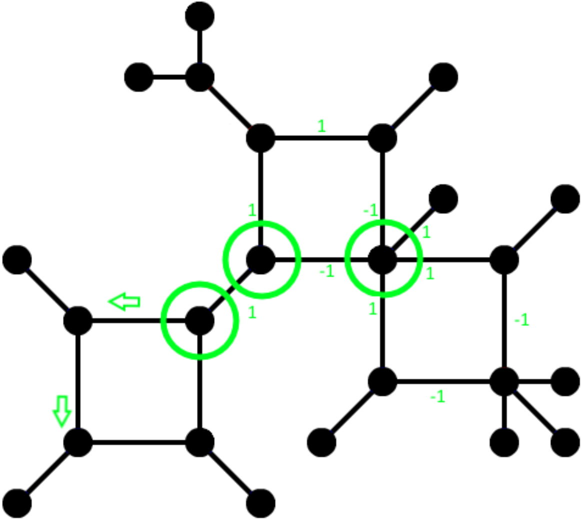

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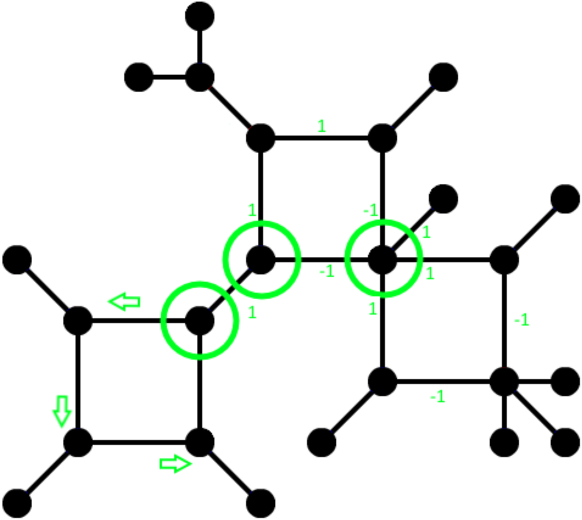

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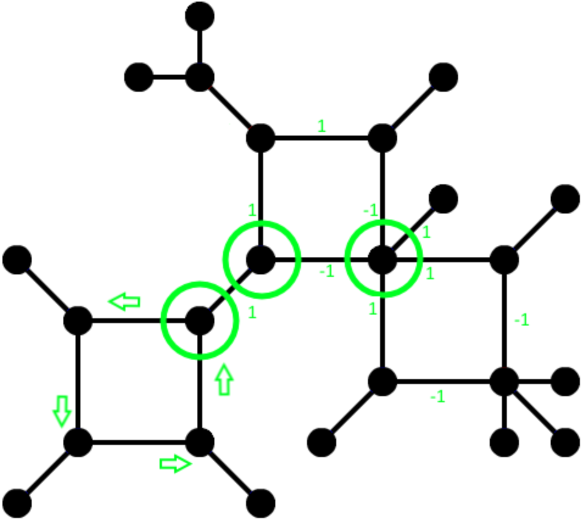

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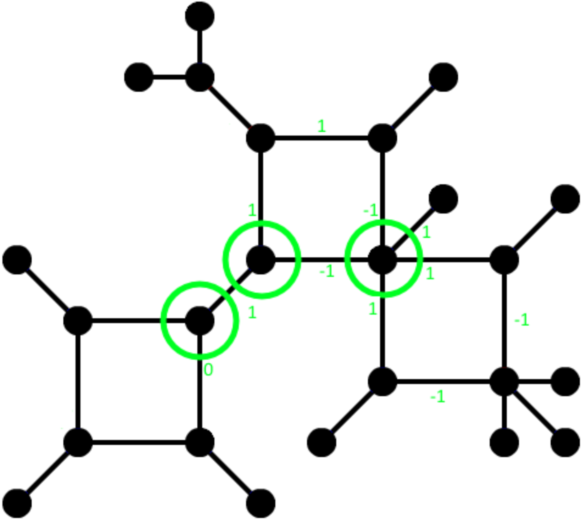

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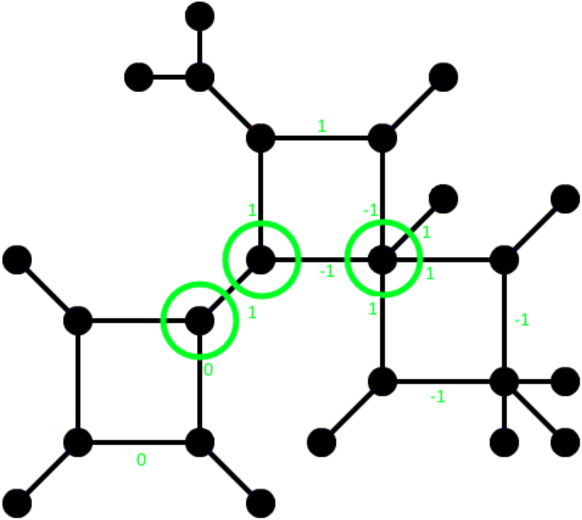

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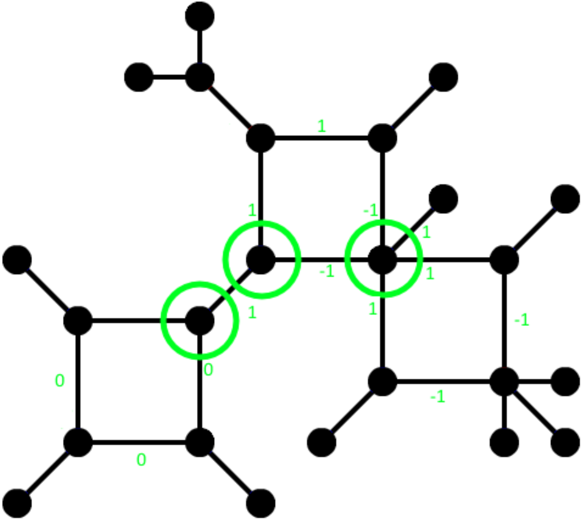

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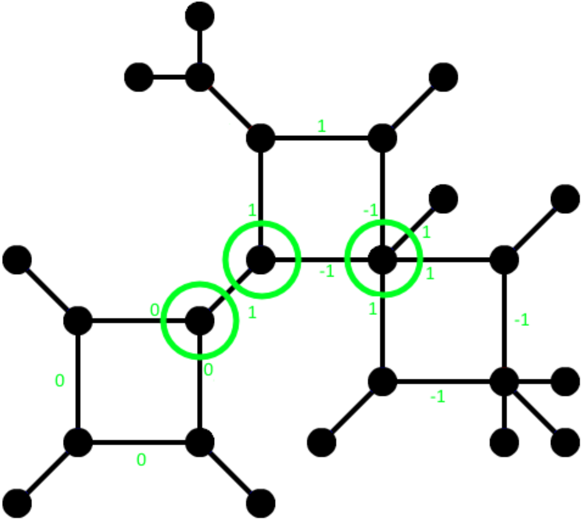

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Richard A. Brualdi, Kathleen P. Kiernan, Seth A. Meyer, Michael W. Schroeder , *Patterns of Alternating Sign Matrices*, Department of Mathematics University of Wisconsin, 2011