REALIZING A FUSION SYSTEM BY A SINGLE FINITE GROUP

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Abstract. We show that every saturated fusion system can be realized as a full subcategory of the fusion system of a finite group. The result suggests the definition of an 'exoticity index' and raises some other questions which we discuss.

1. Introduction

A saturated fusion system $\mathcal{F}$ on a finite p-group S is a category whose objects are the subgroups of S and whose morphisms satisfy certain axioms mimicking the behavior of conjugation maps of finite groups G having S as a Sylow p-subgroup. The axioms of saturated fusion systems were first formulated by Puig in the early 1990s; subsequently the theory of fusion systems (and associated p-local finite groups) drew much attention from homotopy theorists as well as from finite group theorists and representation theorists. We refer the reader to [1] for definitions and some basic properties of saturated fusion systems.

An important feature of fusion systems is that, in them, one sees the action (fusion pattern of finite groups), but not the agent of the action (the finite groups inducing the action). Indeed, there are saturated fusion systems, called exotic fusion systems, which are not fusion systems of any finite groups. On the other hand, Robinson [5] and Leary and Stancu [2] independently showed that every saturated fusion system can be realized as a fusion system of a possibly infinite group.

In [1, §5], while determining the cohomology ring of a p-local finite group with coefficients in $\mathbb{F}_p$, Broto, Levi, and Oliver showed that every saturated fusion system $\mathcal{F}$ on a finite p-group S has a (non-unique) S-S-biset X with certain properties formulated by Linckelmann and Webb which parallel the axioms of saturated fusion systems.(See Theorem 2) Conversely, Puig [3, Ch. 21] and Ragnarsson and Stancu [4] independently showed that given such an S-S-biset X (called by them an $\mathcal{F}$-basic set and a characteristic biset for $\mathcal{F}$, respectively) one can recover the original saturated fusion system $\mathcal{F}$ on S.

In this paper, we point out one significant consequence of Puig's result in [3, Ch. 21] which is apparently not well known in the form that we are going to present, and discuss some questions coming out of it. Let G be a finite group, and let S be a (not necessarily Sylow) p-subgroup of G. We denote by $\mathcal{F}_S(G)$ the fusion system on S such that for every $Q, R \leq S$ we have

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(Q, R) = \{\varphi\colon Q \to R \mid \exists\, x \in G \text{ s.t. } \varphi(u) = xux^{-1}\ \forall\, u \in Q\}.$$

Theorem 1. *For every saturated fusion system $\mathcal{F}$ on a finite p-group S, there is a finite group G having S as a subgroup such that $\mathcal{F} = \mathcal{F}_S(G)$.*

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At a first glance, this theorem seems to contradict the Sylow axiom of saturated fusion systems. But when S_0 is a Sylow p-subgroup of G containing S as a fully $\mathcal{F}_{S_0}(G)$-normalized subgroup (and such an S_0 always exists), what the Sylow axiom requires is exactly that $N_{S_0}(S) = SC_{S_0}(S)$, and this is certainly possible. Note also that the theorem claims that any saturated fusion system can be viewed as a *full* subcategory of a fusion system of some finite group, a stronger statement than the well-known fact that any saturated fusion system is a subcategory of a fusion system of some finite group.

After reviewing the theory of bisets for fusion systems, we give an elementary proof of Theorem 1, and discuss the structure of the group G and the embedding of S into G in §2. Then we raise some questions concerning Theorem 1 and give some partial answers in §3.

2. Bisets and finite groups realizing fusion systems

First let us fix notations. Let S be a group. An S-S-biset is a set with left and right S-action such that $(ux)v = u(xv)$ for $x \in X$, $u, v \in S$. An S-S-biset X can be viewed as an $(S \times S)$-set via $(u, v) \cdot x = uxv^{-1}$ for $x \in X$, $u, v \in S$ and vice versa. From now on, we will view S-S-bisets as $(S \times S)$-sets using this correspondence whenever it is convenient. For a subgroup Q of S and a group homomorphism $\varphi\colon Q \to S$, let

$$S \times_{(Q,\varphi)} S = (S \times S)/\sim$$

where $(xu, y) \sim (x, \varphi(u)y)$ for $x, y \in S$, $u \in Q$, and let $\langle x, y\rangle$ be the $\sim$-equivalence class containing (x, y). One can view this set as an S-S-biset by $t\cdot\langle x, y\rangle = \langle tx, y\rangle$, $\langle x, y\rangle\cdot t = \langle x, yt\rangle$ for $x, y, t \in S$. It is a transitive S-S-biset which is free on the right; it is also free on the left if φ is injective. Viewed as an $(S \times S)$-set, it is isomorphic to

$$(S \times S)/\Delta_Q^{\varphi}$$

where $\Delta_Q^{\varphi} = \{(u, \varphi(u))\colon u \in Q\}$. Finally, for an S-S-biset X, $Q \le S$, and a group homomorphism $\varphi\colon Q \to S$, let ${}_QX$ denote the Q-S-biset obtained from X by restricting the left S-action to Q, and ${}_\varphi X$ the Q-S-biset obtained from X where the left Q-action is induced by φ.

Theorem 2 ([1, 5.5])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then there is a finite S-S-biset X with the following properties:*

1. *Every transitive subbiset of X is isomorphic to $S \times_{(Q,\varphi)} S$ for some $Q \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$.*
2. *For any $Q \le S$ and any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, the Q-S-bisets ${}_QX$ and ${}_\varphi X$ are isomorphic.*
3. *$e(X) := |X|/|S| \not\equiv 0 \mod p$.*

In fact, one can easily show that X must have the same (positive) number of copies of the transitive subbiset $S \times_{(S,\alpha)} S$ for each $\alpha \in \mathrm{Out}_{\mathcal{F}}(S)$. In the case where $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G having S as a Sylow p-subgroup, one can take $X = G$ as an S-S-biset with the action of S given by left and right multiplication in the group G.

Theorem 3 ([3, 21.2, 21.9])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let*

$$X = \bigsqcup_{i=1}^{n} S \times_{(Q_i,\varphi_i)} S$$

be an S-S-biset given by Theorem 2 with $Q_1 = S$, $\varphi_1 = \mathrm{id}_S$. Let $Q \le S$ and let $\varphi\colon Q \to S$ be an injective group homomorphism. Then the following are equivalent:

(1) *φ is a morphism in $\mathcal{F}$.*
(2) *The Q-S-bisets ${}_QX$ and ${}_\varphi X$ are isomorphic.*
(3) *φ is a morphism of $\mathcal{F}_S(G)$, where $G = \mathrm{Aut}({}_1X)$, that is, the group of bijections of the set X preserving the right S-action, and S is identified with a subgroup of G via*

$$\begin{aligned} S &\xrightarrow{\iota} \mathrm{Aut}({}_1X) = G \\ u &\mapsto (x \mapsto ux) \end{aligned}$$

(4) *The fixed-point set $X^{\Delta_Q^\varphi} \neq \emptyset$.*

Proof. (1) $\Rightarrow$ (2): by Theorem 2.

(2) $\Leftrightarrow$ (3): (2) $\Leftrightarrow$ there exists an automorphism $\sigma\colon X \to X$ of a right S-set such that $\sigma(ux) = \varphi(u)\sigma(x)$ for all $x \in X$, $u \in Q$ $\Leftrightarrow$ there exists $\sigma \in G$ such that $\iota(\varphi(u)) = \sigma\iota(u)\sigma^{-1}$ for all $u \in Q$ $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (4): Suppose ${}_QX \cong {}_\varphi X$ as Q-S-bisets. Then we have $X^{\Delta_Q^\varphi} = ({}_QX)^{\Delta_Q^\varphi} \cong ({}_\varphi X)^{\Delta_Q^\varphi} = X^{\Delta_{\varphi(Q)}^{\mathrm{id}_{\varphi(Q)}}}$, and this fixed-point set contains a point $\Delta_S^{\mathrm{id}_S} \in (S \times S)/\Delta_S^{\mathrm{id}_S}$.

(4) $\Rightarrow$ (1): Suppose $X^{\Delta_Q^\varphi} \neq \emptyset$. Then there are some i and $x, y \in S$ such that $(x,y)\Delta_{Q_i}^{\varphi_i} \in X^{\Delta_Q^\varphi}$. This means that $(ux, \varphi(u)y)\Delta_{Q_i}^{\varphi_i} = (x,y)\Delta_{Q_i}^{\varphi_i}$ for all $u \in Q$, or that $\varphi(u) = y\varphi_i(x^{-1}ux)y^{-1}$ for all $u \in Q$. Thus φ belongs to $\mathcal{F}$. $\square$

Now Theorem 1 follows immediately. Let us keep the notations of Theorem 3 and analyze the automorphism group G and the embedding $\iota\colon S \to G$. Since ${}_1X$ is a free right S-set, it is isomorphic to the disjoint union of $e(X)$ copies of the regular right S-set S and hence

$$G \cong S \wr \Sigma_{e(X)}.$$

As we can see, G has a surprisingly simple shape. What does the trick is the way S embeds into G, which encodes fusion data. Let us take a closer look at this embedding. For each i, fix a set $\{t_{ij}\}_{j \in J_i}$ of representatives of the left cosets of Q_i in S. Set $e_i = |S : Q_i| = |J_i|$. Note that $e(X) = \sum_i e_i$. For each $u \in S$, let $\sigma_i(u)$ be the permutation of the index set J_i given by $ut_{ij}Q_i = t_{i\sigma_i(u)(j)}Q_i$. Then we have a decomposition of right S-sets

$$S \times_{(Q_i,\varphi_i)} S = \bigsqcup_{j \in J_i} \langle t_{ij}, S \rangle,$$

where $\langle t_{ij}, S\rangle := \{\langle t_{ij}, x\rangle \mid x \in S\}$ is a regular right S-set. So we have

$$G_i := \mathrm{Aut}({}_1(S \times_{(Q_i,\varphi_i)} S)) \cong S \wr \Sigma_{e_i}.$$

But $S \times_{(Q_i,\varphi_i)} S$ is also a left S-set. Thus $\iota(S)$ is contained in the direct product $\prod_i G_i \le G$. Still writing $\iota\colon S \to \prod_i G_i$ for the induced map and $\pi_i\colon \prod_i G_i \to G_i$ for the canonical projection, we have

$$\pi_i(\iota(u))(\langle t_{ij}, x\rangle) = \langle ut_{ij}, x\rangle = \langle t_{i\sigma_i(u)(j)}, \varphi_i(t_{i\sigma_i(u)(j)}^{-1} ut_{ij})x\rangle$$

for $u, x \in S$. So $\pi_i(\iota(u))$ in G_i viewed as an element of $S \wr \Sigma_{e_i}$ is

$$(\varphi_i(t_{i\sigma_i(u)(j)}^{-1} ut_{ij}); \sigma_i(u))_{j \in J_i}.$$

Note that the component of $\iota(u)$ in the base subgroup $B := S \times \cdots \times S$ ($e(X)$ times) depends on the choice of coset representatives of Q_i's in S, while the component of $\iota(u)$ in the wreathing part $\Sigma_{e(X)}$ does not. Note also that the image $\iota(S)$ of S in $G \cong S \wr \Sigma_{e(X)}$ is not necessarily contained in the base subgroup B. Indeed, since B is normal in G, $S \cap B$ is strongly $\mathcal{F}$-closed, meaning that for every $Q \leq S \cap B$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, we have $\varphi(Q) \leq S \cap B$. So if $\mathcal{F}$ has no strongly closed subgroups, then we have $S \cap B = 1$. Finally, we point out that the group G is in general much larger than is necessary:

Example 4. Let $H = S \rtimes E$, where S is a finite p-group and E is a p'-group of automorphisms of S. Write $e = |E|$. Then one can take $X = H$ as an S-S-biset for the saturated fusion system $\mathcal{F}_S(H)$ with the action of S given by left and right multiplication in the group H. Then we have $G = \mathrm{Aut}({}_1X) = S \wr \Sigma_e$ with

$$\begin{aligned} S &\xrightarrow{\iota} S \wr \Sigma_e \cong G \\ u &\mapsto (\alpha(u); \mathrm{id})_{\alpha \in E}, \end{aligned}$$

and $\mathcal{F}_S(H) = \mathcal{F}_{\iota(S)}(G)$.

3. Some questions and answers

With Theorem 1 in mind, we make the following definition.

Definition 5. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let the *exoticity index* of $\mathcal{F}$ be the minimum of the set

$$\{\log_p |S_0 : S| \mid S \leq S_0 \in \mathrm{Syl}_p(G) \text{ for some finite group } G \text{ such that } \mathcal{F} = \mathcal{F}_S(G)\}.$$

By Theorem 1, the exoticity index of a saturated fusion system $\mathcal{F}$ on a finite p-group S is always a (finite) nonnegative integer, and it is nonzero if and only if $\mathcal{F}$ is exotic. It is natural to ask the following question.

Question 6. *Given an exotic fusion system $\mathcal{F}$ on a finite p-group S, what is the exoticity index of $\mathcal{F}$, and what are the finite groups G achieving the exoticity index?*

The upper bound on the exoticity index given by the construction in Theorem 3 is quite large. Explicitly it is

$$(e(X) - 1) \log_p |S| + \sum_{i \geq 1} \left\lfloor \frac{e(X)}{p^i} \right\rfloor.$$

The following is another natural question, which can be seen as a sort of converse of the previous question.

Question 7. *Given a finite group G with Sylow p-subgroup S and $T \leq S$, when is $\mathcal{F}_T(G)$ saturated?*

Note that by Theorem 1, every saturated fusion system arises in this way. Here we give a criterion in a rather simple case.

Proposition 8. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let $T \leq S$ be strongly $\mathcal{F}$-closed, and let $\mathcal{F}_T$ denote the full subcategory of $\mathcal{F}$ whose objects are the subgroups of T. Then $\mathcal{F}_T$ is a saturated fusion system on T if and only if $S = TC_S(T)$.*

Proof. Clearly $\mathcal{F}_T$ is a fusion system on T. We use Stancu's characterization [6] of saturated fusion systems. Since T is strongly $\mathcal{F}$-closed, it is normal in S, and in particular fully $\mathcal{F}$-normalized. By the Sylow axiom for $\mathcal{F}$, $\mathrm{Aut}_S(T)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(T) = \mathrm{Aut}_{\mathcal{F}_T}(T)$. Thus $\mathrm{Aut}_T(T)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}_T}(T)$ iff $\mathrm{Aut}_T(T) = \mathrm{Aut}_S(T)$ iff $S = TC_S(T)$.

Now we assume $S = TC_S(T)$ and show that $\mathcal{F}_T$ satisfies the extension axiom. Note that for $Q \leq T$, we have $N_S(Q) = N_T(Q)C_S(T)$ and $N_T(Q) \trianglelefteq N_S(Q)$. Thus

$$N_S(Q)/N_T(Q) \cong C_S(T)/(C_S(T) \cap N_T(Q)) = C_S(T)/Z(T),$$

which is independent of Q. Together with the assumption that T is strongly $\mathcal{F}$-closed, this shows that Q is fully $\mathcal{F}_T$-normalized iff Q is fully $\mathcal{F}$-normalized. Thus the extension axiom applied to $\mathcal{F}$ ensures that $\mathcal{F}_T$ satisfies the extension axiom too. □

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