REALIZING FUSION SYSTEMS INSIDE FINITE GROUPS

SEJONG PARK

Abstract. We show that every (not necessarily saturated) fusion system can be realized as a full subcategory of the fusion system of a finite group. This result extends our previous work [5] and complements the related result [4] by Leary and Stancu.

1. Statements of the results

Fix a prime p. Let G be a finite group, and let S be a p-subgroup of G. We denote by $\mathcal{F}_S(G)$ the category whose objects are the subgroups of S and such that for $P, Q \leq S$ we have

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P, Q) = \{\varphi\colon P \to Q \mid \exists\, x \in G \text{ s.t. } \varphi(u) = xux^{-1} \text{ for } u \in P\},$$

where composition of morphisms is composition of functions.

The category $\mathcal{F}_S(G)$ above is a *fusion system* on S. If S is a Sylow p-subgroup of G, then $\mathcal{F}_S(G)$ is *saturated*, but not all saturated fusion systems are of this form. Those saturated fusion systems $\mathcal{F}$ such that $\mathcal{F} \neq \mathcal{F}_S(G)$ for any finite group G having S as a Sylow p-subgroup are called *exotic fusion systems*. We refer the reader to [1] for precise definitions and a general introduction to the subject. In [5] we showed that every saturated fusion system $\mathcal{F}$ on a finite p-group S is of the form $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G having S as a subgroup. The point here is that we are not requiring that S is a Sylow p-subgroup of G. In this short note, we observe that this result holds even when $\mathcal{F}$ is not saturated.

Theorem 1. *Let $\mathcal{F}$ be a fusion system on a finite p-group S. Then there is a finite group G having S as a subgroup such that $\mathcal{F} = \mathcal{F}_S(G)$.*

Thus fusion systems are precisely those categories of the form $\mathcal{F}_S(G)$ for some finite group G and a p-subgroup S of G. Leary and Stancu [4] showed that every fusion system is of the form $\mathcal{F}_S(G)$ where G is a (possibly infinite) group having S as a Sylow p-subgroup, in the sense that every finite p-subgroup of G is conjugate to a subgroup of S. Here $\mathcal{F}_S(G)$ is defined exactly the same way as when G is a finite group. Leary and Stancu's construction uses HNN extensions. As in [5], the proof of Theorem 1 uses a certain S-S-biset associated to the fusion system $\mathcal{F}$, though we use a slightly different one here. We keep the notations of [5].

Definition 2. Let $\mathcal{F}$ be a fusion system on a finite p-group S. A *left semicharacteristic biset* for $\mathcal{F}$ is a finite S-S-biset X satisfying the following properties:

1. X is *$\mathcal{F}$-generated*, i.e., every transitive subbiset of X is of the form $S \times_{(Q,\varphi)} S$ for some $Q \leq S$ and some $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$.
2. X is *left $\mathcal{F}$-stable*, i.e., ${}_Q X \cong {}_\varphi X$ as Q-S-bisets for every $Q \leq S$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$.

A right semicharacteristic biset is defined analogously with right $\mathcal{F}$-stability instead of left $\mathcal{F}$-stability; a semicharacteristic biset is a biset which is both left and right semicharacteristic. When the fusion system is saturated, semicharacteristic bisets are parametrized by Gelvin and Reeh [3] using a result of Reeh [7], and left semicharacteristic bisets can be parametrized analogously. A *left characteristic biset* is a left semicharacteristic biset X such that $|X|/|S| \not\equiv 0 \pmod p$. Broto–Levi–Oliver [2, Proposition 5.5] showed that every saturated fusion system has a left characteristic biset X. In [5], we used this biset X to construct the finite group G in Theorem 1 when $\mathcal{F}$ is saturated. Here we show that every fusion system has a certain left semicharacteristic biset X with an additional property which falls short of making X left characteristic, but which still ensures that the proof in [5] carries over.

Proposition 3. *Every fusion system $\mathcal{F}$ on a finite p-group S has a left semicharacteristic biset X containing $S \times_{(S,\mathrm{id})} S$.*

We are going to prove Proposition 3 and Theorem 1 in the next section.

Remark 4. In [6, Proposition 3.1], a semicharacteristic biset containing $S \times_{(S,\mathrm{id})} S$ is used for a saturated fusion system $\mathcal{F}$ on a finite p-group S. Thus Proposition 3 tells us that [6, Proposition 3.1] holds for an arbitrary fusion system $\mathcal{F}$.

2. Semicharacteristic bisets for fusion systems

Let G be a finite group. A virtual G-set with rational coefficients is an element of the rational Burnside ring $\mathbb{Q} \otimes_{\mathbb{Z}} B(G)$, i.e., a formal sum

$$\sum_H c_H G/H$$

where H runs over a set of representatives of conjugacy classes of subgroups of G and $c_H \in \mathbb{Q}$. If the coefficients of a virtual G-set are all nonnegative integers, then it is simply a (isomorphism class of) finite G-set.

The key step of the proof of Proposition 3 is the following lemma, which says roughly that every virtual S-set with rational coefficients can be stabilized (with respect to a given fusion system $\mathcal{F}$) by adding a virtual S-set with nonnegative rational coefficients.

Lemma 5 (cf. [2, Lemma 5.4])**.** *Let $\mathcal{F}$ be a fusion system on a finite p-group S. Let $\mathcal{H}$ be a collection of subgroups of S which is closed under $\mathcal{F}$-conjugation and taking subgroups. Let X_0 be a virtual S-set with rational coefficients such that $|X_0^P| = |X_0^{P'}|$ for all $P, P' \le S$ with $P, P' \notin \mathcal{H}$ which are $\mathcal{F}$-conjugate. Then there is a virtual S-set X with rational coefficients such that $|X^P| = |X^{P'}|$ for all $P, P' \le S$ which are $\mathcal{F}$-conjugate, $|X^P| = |X_0^P|$ for all $P \le S$ with $P \notin \mathcal{H}$, and $X - X_0$ is a virtual S-set with nonnegative rational coefficients.*

Proof. Consider an $\mathcal{F}$-conjugacy class $\mathcal{P}$ of subgroups of S in $\mathcal{H}$ which are maximal among such subgroups. Choose $P \in \mathcal{P}$ such that $|X_0^P| \ge |X_0^{P'}|$ for all $P' \in \mathcal{P}$. Set

$$X_1 = X_0 + \sum_{P'} \frac{|X_0^P| - |X_0^{P'}|}{|N_S(P')/P'|} S/P',$$

where P' runs over a set of representatives of the subgroups in $\mathcal{P}$ up to S-conjugacy. Then for any $P' \in \mathcal{P}$, we have $|X_1^{P'}| = |X_0^P| = |X_1^P|$. Note that $|X_1^P| = |X_0^P|$ for all $P \leq S$ with $P \notin \mathcal{H}$, and hence $|X_1^P| = |X_1^{P'}|$ for all $P, P' \leq S$ with $P, P' \notin \mathcal{H} \setminus \mathcal{P}$ which are $\mathcal{F}$-conjugate. Also, $X_1 - X_0$ is a virtual S-set with nonnegative rational coefficients. So by repeating this process we get a virtual S-set X with the desired properties. □

Comparing the above lemma to [2, Lemma 5.4], we see that here the lack of saturation is compensated for by allowing rational coefficients.

Proof of Proposition 3. Let

$$Y_0 = \sum_{\alpha \in \mathrm{Out}_{\mathcal{F}}(S)} S \times_{(S,\alpha)} S.$$

Then Y_0 satisfies the assumption of Lemma 5 with respect to the product fusion system $\mathcal{F} \times \mathcal{F}_S(S)$ on $S \times S$ and $\mathcal{H} = \{\Delta(P, \varphi) \mid P < S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)\}$. (See [1, Definition I.6.5, Theorem I.6.6] for the definition and properties of the product fusion system.) Thus Lemma 5 implies that there is a virtual S-set Y with nonnegative rational coefficients which is $\mathcal{F}$-generated and left $\mathcal{F}$-stable and which contains $S \times_{(S,\mathrm{id})} S$. Let m be a large enough positive integer such that $X = mY$ is a (finite) S-set (with nonnegative integer coefficients). Then X is a left semicharacteristic biset for $\mathcal{F}$ containing $S \times_{(S,\mathrm{id})} S$. □

Proof of Theorem 1. Let $\mathcal{F}$ be a fusion system on a finite p-group S and let X be a left semicharacteristic biset for $\mathcal{F}$ containing $S \times_{(S,\mathrm{id})} S$. Let G be the group of automorphisms of X viewed as a right S-set, i.e., the group of bijections $f \colon X \to X$ such that $f(xs) = f(x)s$ for all $x \in X$ and $s \in S$. Then S embeds into G via

$$S \to G, \quad s \mapsto (x \mapsto sx).$$

The proof of [5, Theorem 6] applies verbatim to this situation. Thus we have $\mathcal{F} = \mathcal{F}_S(G)$. □

References

[1] Michael Aschbacher, Radha Kessar, and Bob Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011.

[2] Carles Broto, Ran Levi, and Bob Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), no. 4, 779–856 (electronic).

[3] Matthew Gelvin and Sune Precht Reeh, *Minimal characteristic bisets for fusion systems*, J. Algebra **427** (2015), 345–374.

[4] Ian J. Leary and Radu Stancu, *Realising fusion systems*, Algebra Number Theory **1** (2007), no. 1, 17–34.

[5] Sejong Park, *Realizing a fusion system by a single finite group*, Arch. Math. (Basel) **94** (2010), no. 5, 405–410.

[6] ———, *Counting conjugacy classes of cyclic subgroups for fusion systems*, J. Group Theory **17** (2014), no. 4, 661–666.

[7] Sune Precht Reeh, *The abelian monod of fusion-stable finite sets is free*, preprint (2013).

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland

E-mail address: sejong.park@nuigalway.ie