

CONTROL OF TRANSFER AND WEAK CLOSURE IN FUSION SYSTEMS

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ABSTRACT. We show that K_∞ and K^∞ control transfer in every fusion system on a finite p -group when $p \geq 5$, and that they control weak closure of elements in every fusion system on a finite p -group when $p \geq 3$. This generalizes results of G. Glauberman concerning finite groups.

1. INTRODUCTION

Let G be a finite group with Sylow p -subgroup P . The subgroup $P \cap G'$ of P is called the *focal subgroup of P with respect to G* . It is determined locally by the fusion of elements in P under conjugation by G ; explicitly,

$$P \cap G' = \langle x^{-1}c_g(x) \mid x \in P \text{ and } g \in G \text{ such that } c_g(x) \in P \rangle,$$

where $c_g : G \rightarrow G, x \mapsto g^{-1}xg$. The focal subgroup determines a global property of the group G . Indeed, $P \cap G'$ is a proper subgroup of P if and only if the abelian factor group G/G' has a nontrivial p -subgroup, which is equivalent to saying that G has a nontrivial p -factor group. Also concerned with phenomena of fusion, an element $x \in P$ is said to be *weakly closed in P with respect to H* , for some subgroup H of G containing P , if for every $g \in H$ such that $c_g(x) \in P$ we have $c_g(x) = x$.

In [4, §12–13], Glauberman defines for each finite p -group P , characteristic subgroups $K_\infty(P)$ and $K^\infty(P)$ of P , and shows that, denoting K_∞ or K^∞ by W ,

- (1) when $p \geq 3$, W controls weak closure of elements in P with respect to G , that is, if $x \in P$ is weakly closed in P with respect to $N_G(W(P))$, then x is weakly closed in P with respect to G ;
- (2) when $p \geq 5$, W controls p -transfer in G , that is,

$$P \cap G' = P \cap (N_G(W(P)))'.$$

In this paper, following the strategy of [7] as in our previous work [3], we generalize these results of Glauberman to arbitrary fusion systems:

Theorem 1.1. K_∞ and K^∞ control weak closure of elements in every fusion system on a finite p -group when $p \geq 3$.

Theorem 1.2. K_∞ and K^∞ control transfer in every fusion system on a finite p -group when $p \geq 5$.

As observed by Glauberman, both results fail in general for $p = 2$, as it can be seen in [4, Example 11.3], in the case of the simple group $G = \text{PSL}(2, 17)$. The question about the control of transfer for $p = 3$ is still open (cf. [4, Question 16.3]).

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The above mappings $P \mapsto K_\infty(P)$ and $P \mapsto K^\infty(P)$ are gaining importance within the fusion system context. For instance, in [7], K_∞ and K^∞ play a central role in showing that any $\text{Qd}(p)$ -free fusion system is induced by a finite group. More recently, Robinson [10] uses Theorem 1.2 to obtain results on the number of irreducible characters of height zero in a p -block.

In §2, we define centers and control of weak closure of elements in fusion systems, and prove Theorem 1.1. In §3, we define focal subgroups and control of transfer in fusion systems, and state the main technical theorem (Theorem 3.1) from which Theorem 1.2 follows as a corollary. In §4, we consider the transfer map in fusion systems, and use it to prove some lemmas concerning focal subgroups. In §5, we prove Theorem 3.1. In §6, we generalize additional results on control of transfer and weak closure from [4] to fusion systems. We end this article with a recap in §7 on Glauberman's K_∞ and K^∞ constructions. Our general terminology follows [7] and [3]; in particular, by a fusion system we always mean a saturated fusion system.

2. CONTROL OF WEAK CLOSURE IN FUSION SYSTEMS

Let \mathcal{F} be a fusion system on a finite p -group P . The *center* $Z(\mathcal{F})$ of \mathcal{F} is the largest subgroup Q of P such that every morphism in \mathcal{F} can be extended to a morphism in \mathcal{F} which is the identity map on Q . One can easily show that $Z(\mathcal{F})$ is the set of all *weakly closed elements in P with respect to \mathcal{F}* , i.e. elements $x \in P$ such that $\varphi(x) = x$ for all $\varphi \in \text{Hom}_{\mathcal{F}}(\langle x \rangle, P)$.

Following [7], a *positive characteristic p -functor* is a map W sending every finite p -group Q to a characteristic subgroup $W(Q)$ of Q such that

- (1) $W(Q) \neq 1$ if $Q \neq 1$;
- (2) if $\varphi: Q \rightarrow R$ is an isomorphism of finite p -groups, then $\varphi(W(Q)) = W(R)$.

For a subgroup Q of P , set $W_1(Q) = Q$ and for any positive integer i , define $W_{i+1}(Q) = W(N_P(W_i(Q)))$. We say that Q is *(\mathcal{F}, W) -well-placed* if $W_i(Q)$ is fully \mathcal{F} -normalized for all positive integers i . Note that $W_i(Q) = W(P)$ for all sufficiently large i and that, if Q is (\mathcal{F}, W) -well-placed, so is $W_i(Q)$ for every i . Furthermore, by [3, 2.12], the set of (\mathcal{F}, W) -well-placed subgroups of P forms a conjugation family. Thus, Alperin's fusion theorem implies that every morphism in \mathcal{F} is a composition of a finite number of restrictions of \mathcal{F} -automorphisms of (\mathcal{F}, W) -well-placed subgroups of P .

Suppose further that $Z(Q) \leq W(Q)$ for every finite p -group Q . We say that W *controls weak closure of elements in \mathcal{F}* if

$$Z(N_{\mathcal{F}}(W(P))) = Z(\mathcal{F}).$$

The following proposition shows that control of weak closure in fusion systems is locally determined.

Proposition 2.1. *Let \mathcal{F} be a fusion system on a finite p -group P , and let W be a positive characteristic p -functor such that $Z(Q) \leq W(Q)$ for every finite p -group Q . If there exists $x \in Z(N_{\mathcal{F}}(W(P)))$ such that $x \notin Z(\mathcal{F})$, then there exists an (\mathcal{F}, W) -well-placed subgroup T of P containing x such that $x \in Z(N_{\mathcal{F}}(W(N_P(T))))$ and $x \notin Z(N_{\mathcal{F}}(T))$.*

Proof. We have $Z(N_{\mathcal{F}}(W(P))) \leq Z(P)$, since $W(P)$ is normal in P ; in particular $x \in Z(P)$. By Alperin's fusion theorem, there is an (\mathcal{F}, W) -well-placed subgroup

T of P containing x and a morphism $\varphi \in \text{Aut}_{\mathcal{F}}(T)$ such that $\varphi(x) \neq x$, i.e. $x \notin Z(N_{\mathcal{F}}(T))$. Amongst all such T , choose one with $|N_P(T)|$ maximal. Note that

$$x \in T \cap Z(P) \leq N_P(T) \cap Z(P) \leq Z(N_P(T)) \leq W(N_P(T)).$$

Suppose that $x \notin Z(N_{\mathcal{F}}(W(N_P(T))))$. We have

$$|N_P(T)| \geq |N_P(W(N_P(T)))| \geq |N_P(N_P(T))|,$$

where the first inequality follows from the maximality of $|N_P(T)|$. Hence, T is normal in P and $x \notin Z(N_{\mathcal{F}}(W(N_P(T)))) = Z(N_{\mathcal{F}}(W(P)))$, a contradiction. This shows that $x \in Z(N_{\mathcal{F}}(W(N_P(T))))$. \square

Consider now the positive characteristic p -functors W such that $Z(Q) \leq W(Q)$, for every finite p -group Q . We say that W *satisfies condition (C)* if

$$(C) \quad O_p(G) \cap Z(G) = O_p(G) \cap Z(N_G(W(P))),$$

whenever G is a finite group with Sylow p -subgroup P . Observe that this condition only depends on the subgroup structure in finite groups, and it is sufficient for the proof of the next result.

Theorem 2.2. *Let W be a positive characteristic p -functor such that $Z(Q) \leq W(Q)$ for every finite p -group Q . If W satisfies condition (C), then W controls weak closure of elements in every fusion system \mathcal{F} on a finite p -group P .*

Proof. Suppose that the theorem is false and take a counterexample \mathcal{F} with minimal number $|\mathcal{F}|$ of morphisms. Accordingly, there is an element $x \in Z(N_{\mathcal{F}}(W(P)))$ with $x \notin Z(\mathcal{F})$. By Proposition 2.1, there is an (\mathcal{F}, W) -well-placed subgroup T of P containing x such that $x \in Z(N_{\mathcal{F}}(W(N_P(T))))$ and $x \notin Z(N_{\mathcal{F}}(T))$. If $N_{\mathcal{F}}(T) < \mathcal{F}$, then by the minimality of $|\mathcal{F}|$, we have

$$Z(N_{\mathcal{F}}(W(N_P(T)))) \leq Z(N_{N_{\mathcal{F}}(T)}(W(N_P(T)))) = Z(N_{\mathcal{F}}(T)),$$

contradicting the choice of x . Thus, $1 \neq T \leq O_p(\mathcal{F})$.

Set $Q = O_p(\mathcal{F})$ and $R = QC_P(Q)$. We show that $Q = R$ and hence that Q is \mathcal{F} -centric. Suppose that $Q < R$ and so $N_{\mathcal{F}}(R) < \mathcal{F}$. By the minimality of $|\mathcal{F}|$, we have

$$x \in Z(N_{\mathcal{F}}(W(P))) \leq Z(N_{N_{\mathcal{F}}(R)}(W(P))) = Z(N_{\mathcal{F}}(R)).$$

As $x \in Q$ and every \mathcal{F} -automorphism of Q extends to an \mathcal{F} -automorphism of R (by the extension axiom) this contradicts the assumption that $x \notin Z(\mathcal{F})$. Therefore $Q = R = O_p(\mathcal{F})$ is \mathcal{F} -centric.

By [1, 4.3], there exists a finite group G such that $\mathcal{F} = \mathcal{F}_P(G)$ and such that $O_p(G) = O_p(\mathcal{F})$. By condition (C), we have

$$x \in O_p(G) \cap Z(N_G(W(P))) = O_p(\mathcal{F}) \cap Z(\mathcal{F})$$

and so $x \in Z(\mathcal{F})$, a contradiction. \square

We now show that the positive characteristic p -functors K_{∞} and K^{∞} control weak closure of elements in any fusion system. We refer the reader to Section 7 for the background material. Let us also recall the following standard commutator notation. If H is a subgroup of a group G and $g \in G$, define $[H, g; 0] = H$ and $[H, g; i+1] = [[H, g; i], g]$, for $i \geq 0$.

Proof of Theorem 1.1. Let W denote K_∞ or K^∞ . By Lemma 7.2, $Z(Q) \leq W(Q)$ for every finite p -group Q . Hence, by Theorem 2.2, it will suffice to show that W satisfies condition (C) when $p \geq 3$. Suppose that G is a finite group, P is a Sylow p -subgroup of G . We assume that $W(P) \not\leq G$. Set $Z = Z(O_p(G))$. Let E_0 be the set of all elements $g \in P$ such that $[X, g; p-1] \leq Y$ for every chief factor X/Y of G with $X \leq Z$. Set $E = \langle E_0 \rangle$ and $L = N_G(E)$. By Theorem 7.3, E_0 is nonempty, and, by Theorem 7.4, we have $P \leq L < G$ and $Z \cap Z(G) = Z \cap Z(L)$. Clearly $O_p(G) \leq P \leq L$, so we have $O_p(G) \cap Z(G) = O_p(G) \cap Z(L)$. By induction on the order of G , we have $O_p(L) \cap Z(L) = O_p(L) \cap Z(N_L(W(P)))$. Intersecting both sides with $O_p(G)$, we get

$$O_p(G) \cap Z(L) = O_p(G) \cap Z(N_L(W(P))) \geq O_p(G) \cap Z(N_G(W(P))).$$

Thus, $O_p(G) \cap Z(G) \geq O_p(G) \cap Z(N_G(W(P)))$. The reverse inclusion is trivial. \square

3. FOCAL SUBGROUPS AND CONTROL OF TRANSFER IN FUSION SYSTEMS

Let \mathcal{F} be a fusion system on a finite p -group P . For $Q \leq P$, define

$$[Q, \mathcal{F}] = \langle u^{-1}\varphi(u) \mid u \in Q, \varphi \in \text{Hom}_{\mathcal{F}}(\langle u \rangle, P) \rangle.$$

and call $[P, \mathcal{F}]$ the \mathcal{F} -focal subgroup of P . Note that if $\mathcal{F} = \mathcal{F}_P(G)$ is the fusion system on P defined by the inclusion of P as a Sylow p -subgroup of some finite group G , then the focal subgroup theorem reads ([6, Theorem 7.3.4]),

$$P \cap G' = [P, \mathcal{F}].$$

Given subgroups Q and R of P with $Q \leq R$, we say that Q is *weakly \mathcal{F} -closed in R* if $\varphi(Q) = Q$ for all $\varphi \in \text{Hom}_{\mathcal{F}}(Q, R)$. In particular, if Q is weakly \mathcal{F} -closed in P , then $Q \trianglelefteq P$. For short, and if there is no possible confusion, we simply say that a subgroup Q is weakly \mathcal{F} -closed, instead of weakly \mathcal{F} -closed in P . It is straightforward to show that $[P, \mathcal{F}]$ is weakly \mathcal{F} -closed.

A positive characteristic p -functor W *controls transfer in \mathcal{F}* if the \mathcal{F} -focal subgroup equals the $N_{\mathcal{F}}(W(P))$ -focal subgroup, i.e., if

$$[P, \mathcal{F}] = [P, N_{\mathcal{F}}(W(P))].$$

As for condition (C) in the previous section, we appeal now to a concept which depends only on the subgroup structure in finite groups. Namely, we say that W *satisfies condition (T)* if

$$(T) \quad C_G(O_p(G)) \leq O_p(G) \text{ implies } O_p(G) \cap G' = O_p(G) \cap (N_G(W(P)))',$$

whenever G is a finite group with Sylow p -subgroup P .

Theorem 3.1. *If W is a positive characteristic p -functor satisfying condition (T), then W controls transfer in every fusion system \mathcal{F} on a finite p -group P .*

We prove this theorem in §5 and get Theorem 1.2 as a corollary.

Proof of Theorem 1.2. Let W denote K_∞ or K^∞ . By Theorem 3.1, it will suffice to show that W satisfies condition (T) when $p \geq 5$. Suppose that G is a finite group, P is a Sylow p -subgroup of G , and $C_G(O_p(G)) \leq O_p(G)$. Let $Q = O_p(G)$. We assume that $W(P) \not\leq G$. Let E_0 be the set of all elements $g \in P$ such that $[X, g; p-1] \leq Y$ for every chief factor X/Y of G with $X \leq Q$. Set $E = \langle E_0 \rangle$ and $L = N_G(E)$. By Theorem 7.3, E_0 is nonempty, and, by Theorem 7.4, we have

$P \leq L < G$ and $Q \cap G' = Q \cap L'$. Clearly $Q \leq P \leq L$, and so $Q \leq O_p(L)$; therefore, $C_L(O_p(L)) \leq O_p(L)$. By induction on the order of G , we have

$$Q \cap L' = Q \cap (O_p(L) \cap L') = Q \cap (O_p(L) \cap N_L(W(P))') \leq Q \cap N_G(W(P))'.$$

Thus, $Q \cap G' \leq Q \cap N_G(W(P))'$. Since the opposite containment holds trivially, we get $Q \cap G' = Q \cap N_G(W(P))'$. \square

4. THE TRANSFER MAP IN FUSION SYSTEMS

For a group P , a subgroup Q of P , and a group homomorphism $\varphi: Q \rightarrow P$, let

$$P \times_{(Q, \varphi)} P = P \times P / \sim$$

where $(x, uy) \sim (x\varphi(u), y)$ for $x, y \in P$, $u \in Q$, viewed as a P - P -biset via

$$t \cdot (x, y) = (tx, y) \quad \text{and} \quad (x, y) \cdot t = (x, yt) \quad \text{for } x, y, t \in P.$$

The next theorem plays a crucial role in the theory of fusion systems.

Theorem 4.1 ([2, 5.5]). *Let \mathcal{F} be a fusion system on a finite p -group P . There is a finite P - P -biset X with the following properties:*

- (1) *Every transitive subbiset of X is isomorphic to $P \times_{(Q, \varphi)} P$ for some subgroup Q of P and some group homomorphism $\varphi: Q \rightarrow P$ belonging to \mathcal{F} .*
- (2) *For any $Q \leq P$ and any $\varphi \in \text{Hom}_{\mathcal{F}}(Q, P)$, the Q - P -bisets ${}_Q X$ and ${}_{\varphi} X$ are isomorphic.*
- (3) $|X|/|P| \equiv 1 \pmod{p}$.

Let \mathcal{F} be a fusion system on a finite p -group P . We call a P - P -biset X satisfying the properties of Theorem 4.1 a *P - P -biset associated with \mathcal{F}* . In the case that $\mathcal{F} = \mathcal{F}_P(G)$ is the fusion system defined by a finite group G , there is a suitable non-negative integer k such that the P - P -biset $X = \coprod^k G$ is associated with \mathcal{F} . The integer k is chosen so that $|X|/|P| \equiv 1 \pmod{p}$, the two other conditions of Theorem 4.1 being satisfied by any finite number of copies of the P - P -biset G . We refer the reader to [2, §5] for further details.

Now, suppose that $X = \bigsqcup_i P \times_{(Q_i, \varphi_i)} P$, and let A be an abelian group with trivial P -action. The *transfer map associated with X* is the group homomorphism

$$t_X: H^*(P, A) \rightarrow H^*(P, A)$$

defined by

$$t_X = \sum_i t_{Q_i}^P \circ \text{res}_{\varphi_i},$$

where, for a subgroup Q of P , the map $t_Q^P: H^*(Q, A) \rightarrow H^*(P, A)$ is the transfer. In particular, identifying $H^1(Q, A)$ with the set of group homomorphisms $\text{Hom}(Q, A)$, Theorem VII.3.2 in [6] yields

$$t_Q^P(\alpha)(x) = \sum_{t \in T} \alpha((x \cdot t)^{-1}xt), \quad \text{for all } x \in P \text{ and for all } \alpha \in H^1(Q, A),$$

where T is a set of left coset representatives of Q in P , and where the \cdot symbol denotes the action of P on T induced by the permutation of the cosets. Thus, $x \cdot t \in T$ and $(x \cdot t)^{-1}xt \in Q$.

By [2], we have

$$\text{Im } t_X = H^*(P, A)^{\mathcal{F}} \cong \varprojlim_{\mathcal{F}} H^*(-, A),$$

where $H^*(P, A)^{\mathcal{F}}$ denotes the set of elements $\alpha \in H^*(P, A)$ such that $\text{res}_Q^P(\alpha) = \text{res}_\varphi(\alpha)$ for every $Q \leq P$ and every $\varphi \in \text{Hom}_{\mathcal{F}}(Q, P)$. In particular, if $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G with Sylow p -subgroup P , we have that $\text{Im } t_X = H^*(P, A)^G$ is the set of G -stable elements in $H^*(P, A)$.

The following three lemmas generalize results in [4] to arbitrary fusion systems using the transfer map.

Lemma 4.2 ([4, 4.2]). *Let \mathcal{F} be a fusion system on a finite p -group P and let X be a finite P - P -biset associated with \mathcal{F} . Set $\tau = t_X(\pi)$, where $\pi: P \rightarrow P/P'$ is the canonical surjection, and let $n = |X|/|P|$. If $u \in Z(\mathcal{F})$, then $\tau(u) = u^n P'$.*

Proof. Let $X = \bigsqcup_i P \times_{(Q_i, \varphi_i)} P$. Note that $|X|/|P| = \sum_i |P : Q_i|$. If $u \in Z(\mathcal{F})$, then — switching to multiplicative notation —

$$\begin{aligned} \tau(u) &= \prod_i ((t_{Q_i}^P \circ \text{res}_{\varphi_i})(\pi))(u) \\ &= \prod_i t_{Q_i}^P(\pi \circ \varphi_i)(u) \\ &= \prod_i \prod_{t \in T_i} (\pi \circ \varphi_i)((u \cdot t)^{-1} u t) \end{aligned}$$

where T_i is a set of left coset representatives for Q_i in P . Decompose each T_i into $\langle u \rangle$ -orbits and choose one element t_{ij} from each orbit. Let r_{ij} be the length of the $\langle u \rangle$ -orbit containing t_{ij} . We then obtain $|P : Q_i| = \sum_j r_{ij}$ and since $u^{r_{ij}} \cdot t_{ij} = t_{ij}$

for all i and j , we get

$$\begin{aligned} \tau(u) &= \prod_i \prod_j \prod_{k=0}^{r_{ij}-1} (\pi \circ \varphi_i)((u^{k+1} \cdot t_{ij})^{-1} u (u^k \cdot t_{ij})) \\ &= \prod_i \prod_j (\pi \circ \varphi_i) \left(\prod_{k=0}^{r_{ij}-1} (u^{k+1} \cdot t_{ij})^{-1} u (u^k \cdot t_{ij}) \right) \\ &= \prod_i \prod_j (\pi \circ \varphi_i)(t_{ij}^{-1} u^{r_{ij}} t_{ij}) \\ &= \prod_i \prod_j \pi(u^{r_{ij}}) \\ &= \prod_i \pi(u^{|P:Q_i|}) \\ &= \pi(u^n) = u^n P' \end{aligned}$$

because $u \in Z(\mathcal{F})$. □

Lemma 4.3 ([4, 4.4]). *Let \mathcal{F} be a fusion system on a finite p -group P . Then*

$$[P, \mathcal{F}] \cap Z(\mathcal{F}) = P' \cap Z(\mathcal{F}).$$

Proof. Clearly $P' \cap Z(\mathcal{F}) \leq [P, \mathcal{F}] \cap Z(\mathcal{F})$. Conversely, suppose that $z \in [P, \mathcal{F}] \cap Z(\mathcal{F})$. Let τ be defined as in Lemma 4.2. By Theorem 4.1, for every subgroup Q of P and every morphism $\varphi: Q \rightarrow P$ in \mathcal{F} , we have $\text{res}_Q^P(\tau) = \text{res}_\varphi(\tau)$, that is, $\tau(u) = \tau(\varphi(u))$ for every $u \in Q$. Thus, $\tau(z) = P'$ as $z \in [P, \mathcal{F}]$. On the other hand, $\tau(z) = z^n P'$ by Lemma 4.2. Thus, $z^n \in P'$. Since n is prime to p and z is a p -element, it follows that $z \in P'$. □

Lemma 4.4 ([4, 6.7]). *Let \mathcal{F} be a fusion system on a finite p -group P and let \mathcal{G} be a fusion subsystem of \mathcal{F} on P . Suppose that Q is a subgroup of P which is normal in \mathcal{F} . If $[Q, \mathcal{F}] = [Q, \mathcal{G}]$, then $[P, \mathcal{F}] \cap Q = [P, \mathcal{G}] \cap Q$.*

Proof. Let $R = [P, \mathcal{F}] \cap Q$, $S = [Q, \mathcal{F}] = [Q, \mathcal{G}]$. It will suffice to show that $R \leq [P, \mathcal{G}]$. Clearly, $S \leq R$, and since $Q \triangleleft \mathcal{F}$ and S is weakly \mathcal{F} -closed, we have $S \triangleleft \mathcal{F}$. Furthermore, $R/S \leq Z(\mathcal{F}/S)$, where the quotient fusion system is defined as in [8, 6.2]. In fact, if $x \in R$ and $\overline{\varphi} \in \text{Hom}_{\mathcal{F}/S}(\langle xS \rangle, P/S)$, then $x^{-1}\varphi(x) \in S$ for any $\varphi \in \text{Hom}_{\mathcal{F}}(\langle x \rangle, P)$ inducing $\overline{\varphi}$. This implies that $\overline{\varphi}(xS) = \varphi(x)S = xS$ and so $xS \in Z(R/S)$. By Lemma 4.3, $R/S \leq [P/S, \mathcal{F}/S] \cap Z(\mathcal{F}/S) = (P/S)' \cap Z(\mathcal{F}/S)$. Thus, $R \leq P'S = P'[Q, \mathcal{G}] \leq [P, \mathcal{G}]$. \square

5. PROOF OF THEOREM 3.1

To prove Theorem 3.1, we need the following two lemmas. The first shows that control of transfer in fusion systems is locally determined.

Lemma 5.1. *Let \mathcal{F} be a fusion system on a finite p -group P and let W be a positive characteristic p -functor. If W controls transfer in $N_{\mathcal{F}}(Q)$ for every nontrivial (\mathcal{F}, W) -well-placed subgroup Q of P , then W controls transfer in \mathcal{F} .*

Proof. For every nontrivial (\mathcal{F}, W) -well-placed subgroup Q of P we have

$$\begin{aligned} [N_P(Q), N_{\mathcal{F}}(Q)] &= [N_P(Q), N_{N_{\mathcal{F}}(Q)}(W(N_P(Q)))] \\ &\leq [N_P(W(N_P(Q))), N_{\mathcal{F}}(W(N_P(Q)))] \end{aligned}$$

because $W(N_P(Q)) \trianglelefteq N_P(Q)$. Since $W_i(Q)$ is (\mathcal{F}, W) -well-placed for all i and $W_i(Q) = W(P)$ for all sufficiently large i , we can repeat the above argument until we get

$$[N_P(Q), N_{\mathcal{F}}(Q)] \leq [P, N_{\mathcal{F}}(W(P))].$$

The lemma now follows from Alperin's fusion theorem. \square

The following result is [9, Lemma 3.7], and we include a proof for the convenience of the reader. It considerably shortens Kessar and Linckelmann's proofs of [7, Theorems A and B] (see also [5]).

Lemma 5.2. *Let \mathcal{F} be a fusion system on a finite p -group P . If $Q \triangleleft \mathcal{F}$, then*

$$\mathcal{F} = \langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle$$

where $\langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle$ denotes the subcategory of \mathcal{F} on P generated by $PC_{\mathcal{F}}(Q)$ and $N_{\mathcal{F}}(QC_P(Q))$.

Proof. Let U be a fully \mathcal{F} -normalized centric radical subgroup of P , and take $\varphi \in \text{Aut}_{\mathcal{F}}(U)$. Note that $Q \leq U$ by [1, 1.6]. Since $Q \triangleleft \mathcal{F}$, we have $\theta = \varphi|_Q \in \text{Aut}_{\mathcal{F}}(Q)$. As $UQC_P(Q) \leq N_{\theta}$, there is $\psi \in \text{Hom}_{\mathcal{F}}(UQC_P(Q), P)$ such that $\psi|_Q = \varphi|_Q$. Then

$$\varphi = (\varphi \circ (\psi|_U)^{-1}) \circ \psi|_U.$$

Now, $\varphi \circ (\psi|_U)^{-1}$ is a morphism in $PC_{\mathcal{F}}(Q)$ and $\psi|_U$ is a morphism in $N_{\mathcal{F}}(QC_P(Q))$ because $\psi(QC_P(Q)) = QC_P(Q)$. Consequently, we have that φ is a morphism in $\langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle$. By Alperin's fusion theorem, it follows that $\mathcal{F} = \langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle$. \square

Now we prove Theorem 3.1. The proof follows exactly the line of arguments in the proof of [7, Theorem B]. It also incorporates arguments in [4, 6.8], generalized to arbitrary fusion systems, if needed, as in Lemma 4.4.

Proof of Theorem 3.1. Suppose that the theorem is false and take a counterexample \mathcal{F} with a minimal number $|\mathcal{F}|$ of morphisms.

- $O_p(\mathcal{F}) \neq 1$: Set $Q = O_p(\mathcal{F})$. If $Q = 1$, then for any nontrivial (\mathcal{F}, W) -well-placed subgroup T of P , we have $N_{\mathcal{F}}(T) < \mathcal{F}$. By the minimality of $|\mathcal{F}|$, it follows that W controls transfer in $N_{\mathcal{F}}(T)$. Now, Lemma 5.1 implies that W controls transfer in \mathcal{F} , a contradiction.

- $QC_P(Q) > Q$: Set $R = QC_P(Q)$. If $R = Q$, then by [1, 4.3], there exists a finite group G such that $\mathcal{F} = \mathcal{F}_P(G)$ and $C_G(Q) \leq Q$; in particular, $O_p(G) = Q$. By condition (T), we have

$$(*) \quad Q \cap G' = Q \cap N_G(W(P))'.$$

For every subgroup H of G , let $\overline{H} = HQ/Q$. Since $|\mathcal{F}_{\overline{P}}(\overline{G})| < |\mathcal{F}|$, we have $[\overline{P}, \mathcal{F}_{\overline{P}}(\overline{G})] = [\overline{P}, N_{\mathcal{F}_{\overline{P}}(\overline{G})}(W(\overline{P}))]$, i.e., $\overline{P} \cap \overline{G}' = \overline{P} \cap N_{\overline{G}}(W(\overline{P}))'$. If L is the subgroup of G containing Q such that $\overline{L} = N_{\overline{G}}(W(\overline{P}))$, then $\overline{P} \cap \overline{G}' = \overline{P} \cap \overline{L}'$ and so $P \cap G' \leq L'Q$. This gives

$$P \cap G' \leq P \cap L'Q = (P \cap L')Q$$

by Dedekind's lemma (see [4, 6.2]). Letting $T_1 = P \cap G'$, $T_2 = P \cap L'$, we obtain, again by Dedekind's lemma,

$$P \cap G' = QT_2 \cap T_1 = (Q \cap T_1)T_2 = (Q \cap G')(P \cap L').$$

Hence, the containment $N_G(W(P)) \leq L$ implies

$$P \cap G' = (Q \cap L')(P \cap L') = P \cap L'.$$

Since $W(\overline{P})$ is characterisitic in \overline{P} , we have $\overline{P} \leq \overline{L}$. As $O_p(\overline{G}) = 1$, we have $\overline{L} < \overline{G}$, whence $P \leq L < G$. Clearly $Q \trianglelefteq L$ and $C_L(Q) \leq C_G(Q) \leq Q$. By the uniqueness of G , it follows that $\mathcal{F}_P(L) < \mathcal{F}$. Thus, the minimality of $|\mathcal{F}|$ and the focal subgroup theorem imply

$$P \cap L' = [P, \mathcal{F}_P(L)] = [P, N_{\mathcal{F}_P(L)}(W(P))] \leq [P, N_{\mathcal{F}}(W(P))].$$

Therefore, $[P, \mathcal{F}] = P \cap G' = P \cap L' = [P, N_{\mathcal{F}}(W(P))]$, a contradiction.

- $\mathcal{F} = PC_{\mathcal{F}}(Q)$: Suppose that $PC_{\mathcal{F}}(Q) \neq \mathcal{F}$. By the minimality of $|\mathcal{F}|$, W controls transfer in $PC_{\mathcal{F}}(Q)$. On the other hand, $N_{\mathcal{F}}(R) \neq \mathcal{F}$ because $R > Q = O_p(\mathcal{F})$, and hence W also controls transfer in $N_{\mathcal{F}}(R)$. By Lemma 5.2, it follows that W controls transfer in \mathcal{F} , a contradiction. Thus, we have $\mathcal{F} = PC_{\mathcal{F}}(Q)$.

- Now let V be the inverse image of $W(P/Q)$ in P under the canonical surjection. By [7, 3.4], we have $N_{\mathcal{F}}(V)/Q = N_{\mathcal{F}/Q}(W(P/Q))$. By the minimality of $|\mathcal{F}|$, we have $[P/Q, \mathcal{F}/Q] = [P/Q, N_{\mathcal{F}}(V)/Q]$, and so

$$[P, \mathcal{F}]/([P, \mathcal{F}] \cap Q) = [P, N_{\mathcal{F}}(V)]/([P, N_{\mathcal{F}}(V)] \cap Q).$$

Since $\mathcal{F} = PC_{\mathcal{F}}(Q)$ and $V \trianglelefteq P$, we have $[Q, \mathcal{F}] = [Q, P] = [Q, N_{\mathcal{F}}(V)]$. By Lemma 4.4, we have $[P, \mathcal{F}] \cap Q = [P, N_{\mathcal{F}}(V)] \cap Q$. Thus, $[P, \mathcal{F}] = [P, N_{\mathcal{F}}(V)]$.

Since $W(P/Q) \neq 1$, we have $Q < V$ and so $N_{\mathcal{F}}(V) < \mathcal{F}$. By the minimality of $|\mathcal{F}|$, it follows that $[P, N_{\mathcal{F}}(V)] = [P, N_{N_{\mathcal{F}}(V)}(W(P))] \leq [P, N_{\mathcal{F}}(W(P))]$. This shows that $[P, \mathcal{F}] = [P, N_{\mathcal{F}}(W(P))]$, a contradiction. \square

6. ADDITIONAL RESULTS

In this section, we prove some additional results on control of transfer and weak closure that generalize the statements [4, 6.3, 12.5 and 12.8]:

Proposition 6.1 ([4, 6.9]). *Suppose that \mathcal{F} is a fusion system on a nontrivial finite p -group P such that $\text{Aut}_{\mathcal{F}}(P)$ is a p -group (or, equivalently, $N_{\mathcal{F}}(P) = \mathcal{F}_P(P)$). If there exists a positive characteristic p -functor that controls transfer in \mathcal{F} and every quotient of \mathcal{F} , then $[P, \mathcal{F}] < P$.*

Proof. Suppose that the proposition is false and take a counterexample \mathcal{F} with a minimal number $|\mathcal{F}|$ of morphisms. If W is a positive characteristic p -functor that controls transfer in \mathcal{F} , then $[P, \mathcal{F}] = [P, N_{\mathcal{F}}(W(P))]$ and so we must have $W(P) \triangleleft \mathcal{F}$. Set $Z = Z(W(P))$, $\overline{P} = P/Z$, and $\overline{\mathcal{F}} = \mathcal{F}/Z$, where the quotient fusion system is defined as in [8, 6.2]. If $Z < P$, then $\overline{P} \neq 1$, $|\overline{\mathcal{F}}| < |\mathcal{F}|$, and $\text{Aut}_{\overline{\mathcal{F}}}(\overline{P})$ is a p -group because it is a homomorphic image of $\text{Aut}_{\mathcal{F}}(P)$. By the minimality of $|\mathcal{F}|$, we have $[\overline{P}, \overline{\mathcal{F}}] = [\overline{P}, \overline{\mathcal{F}}] < \overline{P}$, contradicting $[P, \mathcal{F}] = P$. So $Z = P$ and hence P is abelian. By Burnside's theorem (see [8, Theorem 3.8]), $\mathcal{F} = N_{\mathcal{F}}(P) = \mathcal{F}_P(P)$ and so $1 = [P, P] = [P, \mathcal{F}] = P \neq 1$, a contradiction. \square

Corollary 6.2 is now a consequence of Theorem 1.2 and the preceding proposition.

Corollary 6.2 ([4, 12.5]). *Let $p \geq 5$. If \mathcal{F} is a fusion system on a nontrivial finite p -group P such that $\text{Aut}_{\mathcal{F}}(P)$ is a p -group, then $[P, \mathcal{F}] < P$.*

Proposition 6.3 ([4, 7.9]). *Let W be a positive characteristic p -functor such that $Z(Q) \leq W(Q)$ for every finite p -group Q . Suppose that, whenever G is a finite group with Sylow p -subgroup P such that $W(P) \ntriangleleft G$, there is $g \in P - O_p(G)$ such that $[Z(O_p(G)), g, g] = 1$. If \mathcal{F} is a fusion system on a finite p -group P , then*

$$Z(P)^p \cap Z(N_{\mathcal{F}}(W(P))) \leq Z(\mathcal{F}).$$

Proof. Suppose the proposition is false and take a counterexample \mathcal{F} with a minimal number $|\mathcal{F}|$ of morphisms. This gives an element $x \in Z(P)^p \cap Z(N_{\mathcal{F}}(W(P)))$ such that $x \notin Z(\mathcal{F})$. By Proposition 2.1, there is an (\mathcal{F}, W) -well-placed subgroup T of P containing x such that $x \in Z(N_{\mathcal{F}}(W(N_P(T))))$ and $x \notin Z(N_{\mathcal{F}}(T))$. As $x \in Z(P)^p \leq Z(N_P(T))^p$, the minimality of \mathcal{F} implies $\mathcal{F} = N_{\mathcal{F}}(T)$; in particular, $x \in T \leq O_p(\mathcal{F})$. Let $Q = O_p(\mathcal{F})$. By Lemma 5.2, we have $\mathcal{F} = \langle PC_{\mathcal{F}}(Q), N_{\mathcal{F}}(QC_P(Q)) \rangle$. Since $x \in Q \cap Z(P)$, we have $x \in Z(PC_{\mathcal{F}}(Q))$ and hence $x \notin Z(N_{\mathcal{F}}(C_P(Q)))$. On the other hand,

$$x \in Z(P)^p \cap Z(N_{N_{\mathcal{F}}(C_P(Q))}(W(P)))$$

and so by the minimality of \mathcal{F} , we have $\mathcal{F} = N_{\mathcal{F}}(QC_P(Q))$. Therefore, $Q = QC_P(Q)$ is \mathcal{F} -centric and hence \mathcal{F} is constrained. By [1, 4.3], there exists a finite group G with Sylow p -subgroup P such that $\mathcal{F} = \mathcal{F}_P(G)$ and the result now follows from [4, Theorem 7.9]. \square

As a special case of Proposition 6.3 we get (utilizing [4, 12.3]) the following corollary.

Corollary 6.4 ([4, 12.8]). *Let W denote either K^∞ or K_∞ . If \mathcal{F} is a fusion system on a finite p -group P , then $Z(P)^p \cap Z(N_{\mathcal{F}}(W(P))) \leq Z(\mathcal{F})$.*

7. APPENDIX : K^∞ AND K_∞

For the sake of completeness, we summarize the definitions and properties of the positive characteristic p -functors K^∞ and K_∞ , as introduced in [4, § 12 and 13].

Let P be a finite p -group and let $Q \leq P$. Define $\mathcal{M}(P; Q)$ to be the set of subgroups B of P normalized by Q and such that $B/Z(B)$ is abelian. We identify two useful subsets. First, $\mathcal{M}^*(P; Q)$ will denote the subset of $\mathcal{M}(P; Q)$ containing those subgroups B for which the conjugation action of Q on B induces the trivial action on $B/Z(B)$. The second subset, $\mathcal{M}_*(P; Q)$, is slightly more complicated. For this subset, we choose those subgroups B of $\mathcal{M}(P; Q)$ satisfying the following condition: if $A \in \mathcal{M}(P; B)$ such that $A \leq Q \cap C_P([Z(B), A])$ and $A' \leq C_P(B)$, then the conjugation action of A on B induces the trivial action on $B/Z(B)$.

Set $K_{-1}(P) = P$, and for $i \geq 0$, define inductively

$$K_i(P) = \begin{cases} \langle \mathcal{M}^*(P; K_{i-1}(P)) \rangle & \text{for } i \text{ odd} \\ \langle \mathcal{M}_*(P; K_{i-1}(P)) \rangle & \text{for } i \text{ even.} \end{cases}$$

Definition 7.1. Let P be a finite p -group. We set

$$K^\infty(P) = \bigcap_{i \geq -1, \text{ odd}} K_i(P)$$

$$K_\infty(P) = \langle K_i(P) \mid i \geq 0, \text{ even} \rangle.$$

Here are the main properties of $K^\infty(P)$ and $K_\infty(P)$.

Lemma 7.2. [4, 13.1] *Let P be a finite p -group and let W denote either K^∞ or K_∞ .*

- (1) $W(P)$ is a characteristic subgroup of P .
- (2) $W(P)$ contains $Z(P)$. In particular, if $P \neq 1$, then $W(P) \neq 1$.
- (3) If $\varphi : P \rightarrow Q$ is a group isomorphism, then $\varphi(W(P)) = W(Q)$.

Consequently, the mappings $P \mapsto K_\infty(P)$ and $P \mapsto K^\infty(P)$ are positive characteristic p -functors.

The next theorem is the key result — for our purposes — of Glauberman on the K -infinity subgroups.

Theorem 7.3. [4, 12.3] *Let G be a finite group with Sylow p -subgroup P and set $T = O_p(G)$. If $K^\infty(P)$ or $K_\infty(P)$ is not normal in G , then there exists $g \in P$, with $g \notin O_p(G)$ such that:*

- (1) $[X, g; 4] \leq Y$ for every chief factor X/Y of G such that $X \leq T$;
- (2) $[Z(T), g, g] = 1$.

We also need the following result.

Theorem 7.4. [4, 7.2, 7.3] *Let G be a finite group with Sylow p -subgroup P and suppose that N is a normal p -subgroup of G and E_0 is a nonempty subset of P . Assume that:*

- (1) $\langle E_0 \rangle^g = \langle E_0 \rangle$ whenever $g \in G$ and $\langle E_0 \rangle^g \leq P$; and
- (2) $[X, g; p-1] \leq Y$ for every $g \in E_0$ and every chief factor X/Y of G such that $X \leq N$;

Let $E = \langle E_0 \rangle$ and $L = N_G(E)$. Then $P \leq L$, $N \cap Z(G) = N \cap Z(L)$, $N \cap G' = N \cap L'$, and $[N, G] = [N, L]$.

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