The Weighted Fusion Category Algebra

A thesis presented for the degree of
Doctor of Philosophy
at the University of Aberdeen

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JUNE 2008

Declaration

I hereby declare that this thesis has been composed by me and is based on work done by me and that this thesis has not been presented for assessment in any previous application for a degree, diploma or other similar award. I also declare that all sources of information have been specifically acknowledged and all quotations distinguished by quotation marks.

Summary

In 1986, Alperin [**2**] proposed his conjecture that the number $\ell(b)$ of isomorphism classes of simple kG-modules in a p-block b of a finite group G (k a field of characteristic p) is equal to the number of conjugacy classes of weights of G in b, which are pairs (R, V) of p-subgroups R of G and projective simple modules V of $kN_G(R)/R$ lying in $\mathrm{Br}_R(b)$ as $kN_G(R)$-modules. Due to its preciseness in predicting the p-local determination of the global invariant $\ell(b)$ of the block b and its unifying perspective, Alperin's weight conjecture has drawn enormous interests in modular representation theory of finite groups. Besides being confirmed for specific types of groups including symmetric groups and finite general linear groups by Alperin and Fong [**4**] and An [**5**], and p-solvable groups by Isaacs and Navarro [**21**], there are various reformulations of Alperin's weight conjecture, notably by Knörr and Robinson [**23**] and Dade [**10**] [**11**].

In this thesis, we are concerned with yet another such reformulation by Linckelmann based on fusion systems. In [**26**], Linckelmann defines the weighted fusion category algebra $\overline{\mathcal{F}}(b)$ of a p-block b of a finite group G as a certain subalgebra of the twisted category algebra $k_\alpha\overline{\mathcal{F}}^c$ of some modification $\overline{\mathcal{F}}^c$ of the fusion system $\mathcal{F}$ of the block b, up to the conjectural existence and uniqueness of $\alpha \in H^2(\overline{\mathcal{F}}^c, \underline{k}^\times)$, and shows that Alperin's weight conjecture for the block b is equivalent to the equality between the number of isomorphism classes of simple kGb-modules and that of simple $\overline{\mathcal{F}}(b)$-modules, and that the weighted fusion category algebra $\overline{\mathcal{F}}(b)$ is always quasi-hereditary.

Certainly we want to compute this weighted fusion category algebra explicitly at least for some cases. Also there arise some obvious questions: Can we say more about the structure of the weighted fusion category algebras? How is it related to some other constructions around genuine groups such as q-Schur algebras? What does it bear on the original Alperin's weight conjecture?

In Chapter 1, we review the definition and properties of fusion systems due to Puig and show that fusion systems of finite groups and its blocks are all special cases of this general notion of fusion systems. To do this, we develop necessary block theoretic machinary from the first principle. Finally we state Alperin's weight conjecture and reformulate it in terms of fusion systems.

In Chapter 2, we define the weighted fusion category algebra $\overline{\mathcal{F}}(b)$ for a block b of a finite group following Linckelmann and analyze its quiver. This analysis gives an alternative module theoretic proof that $\overline{\mathcal{F}}(b)$ is quasi-hereditary, and moreover shows that the weighted fusion category algebra $\overline{\mathcal{F}}(b)$ belongs to a special type of quasi-hereditary algebras. We further investigate some consequences of this new observation, and compute the Morita types of the weighted fusion category algebras of all tame blocks.

Finally, in Chapter 3, we compute the structure of the weighted fusion category algebra $\overline{\mathcal{F}}(b_0)$ for the principal 2-block b_0 of $\mathrm{GL}_n(q)$ for small n and compare them with those of the q-Schur algebras $\mathcal{S}_n(q)$, another quasi-hereditary algebra canonically associated with $\mathrm{GL}_n(q)$ possessing representation theoretic information of $\mathrm{GL}_n(q)$. It turns out that $\overline{\mathcal{F}}(b_0)$ is the quotient of $\mathcal{S}_n(q)$ by its socle when $n = 2$, and they are involved in a certain pull back diagram when $n = 3$. This result is interesting because the q-Schur algebra is not defined in terms of p-local structure of $\mathrm{GL}_n(q)$. Moreover, as a consequence we get a canonical bijection between simple $k\mathrm{GL}_2(q)b_0$-modules and weights for b_0, which gives some hint of finding a canonical bijection from the "numerical coincidence" of Alperin's weight conjecture and a possible structural understanding.

Degree of Doctor of Philosophy
University of Aberdeen

Abstract of Thesis

Title	**The Weighted Fusion Category Algebra**
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Date	**June 2008**

We investigate the weighted fusion category algebra $\overline{\mathcal{F}}(b)$ of a block b of a finite group, which is defined by Markus Linckelmann based on the fusion system of the block b to reformulate Alperin's weight conjecture. We present the definition and fundamental properties of the weighted fusion category algebras from the first principle. In particular, we give an alternative proof that they are quasi-hereditary, and show that they are Morita equivalent to their Ringel duals. We compute the structure of the weighted fusion category algebras of tame blocks and principal 2-blocks of $\mathrm{GL}_n(q)$ explicitly in terms of their quivers with relations, and compare them with that of the q-Schur algebras $\mathcal{S}_n(q)$ for q odd prime powers and $n = 2, 3$. As a result, we find structural connections between them.

Keywords: fusion system, Alperin's weight conjecture, weighted fusion category algebra, quasi-hereditary algebra, q-Schur algebra.

Contents

CHAPTER 1

Fusion Systems and Alperin's Weight Conjecture

Fusion is a generic term used to describe conjugacy relations in groups. More precisely, two subgroups Q and R of a Sylow p-subgroup P of a finite group G is said to be *fused* by G if they are not conjugate in P but are conjugate in G. Alperin's fundamental theorem [**1**] shows that fusion in a finite group G is completely determined "p-locally", namely by the normalizers of nontrivial p-subgroups of G, so called *p-local subgroups* of G. Classical theorems of Burnside, Frobenius and Grün on the existence of nontrivial p-factor groups can be systematically derived from the local analysis of fusion in a given group.

Using category theoretic language, we can define the fusion system of a group G on its Sylow p-subgroup P as a category of subgroups of P with morphisms given by "conjugations in G". Furthermore, Alperin and Broué [**3**] showed that one can extend this notion of fusion systems to p-blocks of finite groups and their defect groups using the notion of Brauer pairs. Puig further generalized fusion systems of blocks to give a current definition of fusion systems by axiomatizing the essential properties of fusion systems of finite groups and blocks.

On the other hand, in 1986, Alperin [**2**] proposed his conjecture that the number $\ell(B)$ of isomorphism classes of simple kG-modules in a block B of kG is equal to the number of conjugacy classes of weights of G in B. Weights are pairs (R, V) of p-subgroups R of G and projective simple modules V of $kN_G(R)/R$, both of which are defined in terms of p-local subgroups of G, i.e. the normalizers of nontrivial p-subgroups of G, except when $R = \{1\}$. As simple kG-modules, each weight of G belong to a unique block of kG. The main point of Alperin's weight conjecture is that $\ell(B)$ is determined "p-locally" in a precisely described manner.

In this chapter we review the definition of fusion systems and show that fusion systems of groups and blocks are special cases of this definition. Then we reformulate Alperin's weight conjecture using fusion systems.

For general reference on modular representation theory, we use [**35**]. We refer to [**17**] for fusion systems and related category theoretic constructions, in particular [**27**] [**22**] [**36**].

1. Fusion Systems

Throughout this paper, p is a prime.

1.1. Definition of the Fusion System. Let G be a group, Q, R, H be subgroups of G, and $x \in G$. Let $c_x \colon G \to G$ denote the conjugation map by x, which is defined by

$$c_x(u) = xux^{-1}, \qquad u \in G.$$

Let ${}^xQ = c_x(Q)$. Let

$$\mathrm{Hom}_H(Q, R) = \{\, \varphi \colon Q \to R \mid \varphi = c_x|_Q \text{ for some } x \in H \,\},$$
$$\mathrm{Aut}_H(Q) = \mathrm{Hom}_H(Q, Q).$$

We write $Q \leq R$ or $R \geq Q$ when Q is a subgroup of R; we write $Q < R$ or $R > Q$ when Q is a proper subgroup of R.

DEFINITION 1.1. A *category on a finite p-group P* is a category $\mathcal{F}$ whose object set is the set of all subgroups of P, and for each pair Q, R of subgroups of P, whose morphism set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ is a set of injective group homomorphisms from Q to R, where the composition of morphisms is the usual composition of maps, satisfying the following properties:

(1) if Q, R are subgroups of P such that $Q \subseteq R$, then the inclusion map $Q \hookrightarrow R$ from Q to R is a morphism in $\mathcal{F}$;
(2) if $\varphi \colon Q \to R$ is a morphism in $\mathcal{F}$, so are the induced group isomorphism $\varphi \colon Q \xrightarrow{\cong} \varphi(Q)$ and its inverse $\varphi^{-1} \colon \varphi(Q) \xrightarrow{\cong} Q$.

We write $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Hom}_{\mathcal{F}}(Q, Q)$ for a subgroup Q of P.

DEFINITION 1.2. Let $\mathcal{F}$ be a category on a finite p-group P and let Q be a subgroup of P.

(1) Q is said to be *fully $\mathcal{F}$-normalized* if $|N_P(Q)| \geq |N_P(Q')|$ for all Q' which is isomorphic to Q in $\mathcal{F}$.
(2) Q is said to be *fully $\mathcal{F}$-centralized* if $|C_P(Q)| \geq |C_P(Q')|$ for all Q' which is isomorphic to Q in $\mathcal{F}$.

DEFINITION 1.3. Let $\mathcal{F}$ be a category on a finite p-group P and let $\varphi \colon Q \to R$ be a morphism in $\mathcal{F}$. We set

$$N_\varphi = \{\, y \in N_P(Q) \mid \varphi \circ c_y|_Q \circ \varphi^{-1} \in \mathrm{Aut}_P(\varphi(Q)) \,\}.$$

DEFINITION 1.4. A *fusion system on a finite p-group P* is a category $\mathcal{F}$ on P such that

(1) $\mathrm{Hom}_P(Q, R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q, R)$ for all subgroups Q, R of P;
(2) (Sylow axiom) $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$;
(3) (Extension axiom) for every morphism $\varphi\colon Q \to P$ in $\mathcal{F}$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalized, there is a morphism $\psi\colon N_\varphi \to P$ in $\mathcal{F}$ such that $\psi|_Q = \varphi$.

Fusion systems were originally defined by Puig; the above definition is equivalent to Puig's original definition and appears in Stancu [**33**]. In Theorem 1.8, we show that this definition is equivalent to that of Broto, Levi and Oliver [**8**].

1.2. Properties of Fusion Systems.

PROPOSITION 1.5. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. A subgroup Q of P is fully $\mathcal{F}$-normalized if and only if Q is fully $\mathcal{F}$-centralized and* $\mathrm{Aut}_P(Q)$ *is a Sylow p-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(Q)$.

PROOF. Suppose that Q is a fully $\mathcal{F}$-centralized subgroup of P and $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. From the isomorphism of groups $\mathrm{Aut}_P(Q) \cong N_P(Q)/C_P(Q)$, we have the identity

$$|N_P(Q)| = |\mathrm{Aut}_P(Q)| \cdot |C_P(Q)|.$$

Let Q' be a subgroup of P which is isomorphic to Q in $\mathcal{F}$. Then we also have

$$|N_P(Q')| = |\mathrm{Aut}_P(Q')| \cdot |C_P(Q')|.$$

Now we have $|C_P(Q)| \geq |C_P(Q')|$ because Q is fully $\mathcal{F}$-centralized. Also we have $|\mathrm{Aut}_P(Q)| \geq |\mathrm{Aut}_P(Q')|$ because $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $Aut_{\mathcal{F}}(Q)$, $\mathrm{Aut}_P(Q')$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q')$, and $\mathrm{Aut}_{\mathcal{F}}(Q) \cong \mathrm{Aut}_{\mathcal{F}}(Q')$. Thus $|N_P(Q)| \geq |N_P(Q')|$ and it follows that Q is fully $\mathcal{F}$-normalized.

Conversely, suppose that Q is a fully $\mathcal{F}$-normalized subgroup of P. Let $\varphi\colon Q' \to Q$ be any isomorphism in $\mathcal{F}$ onto Q. By the extension axiom, there is a morphism $\psi\colon N_\varphi \to P$ in $\mathcal{F}$ such that $\psi|_{Q'} = \varphi$. Since $Q'C_P(Q') \subseteq N_\varphi$, we have $\psi(C_P(Q')) \leq C_P(Q)$, so $|C_P(Q')| \leq |C_P(Q)|$. Thus Q is a fully $\mathcal{F}$-centralized.

Now let us show that $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Suppose not; let Q be a fully $\mathcal{F}$-normalized subgroup of P which is maximal subject to the property that $\mathrm{Aut}_P(Q)$ is not a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. By the Sylow axiom, $Q < P$. Choose a p-subgroup R of $\mathrm{Aut}_{\mathcal{F}}(Q)$ which contains $\mathrm{Aut}_P(Q)$ as a proper normal subgroup. Choose $\varphi \in R - \mathrm{Aut}_P(Q)$. Then $\varphi \mathrm{Aut}_P(Q)\varphi^{-1} = \mathrm{Aut}_P(Q)$, so $N_\varphi = N_P(Q)$. Then by the extension axiom there exists $\psi \in \mathrm{Aut}_{\mathcal{F}}(N_P(Q))$

such that $\psi|_Q = \varphi$. Upon replacing ψ with its p-part, we may assume that ψ is a p-element of $\mathrm{Aut}_{\mathcal{F}}(N_P(Q))$. Let $\sigma\colon N_P(Q) \to P$ be a morphism in $\mathcal{F}$ such that $\sigma N_P(Q)$ is fully $\mathcal{F}$-normalized. Since $N_P(Q) > Q$, the maximality assumption on Q implies that $\mathrm{Aut}_P(\sigma N_P(Q))$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(\sigma N_P(Q))$. Note that $\sigma\psi\sigma^{-1}$ is a p-element of $\mathrm{Aut}_{\mathcal{F}}(\sigma N_P(Q))$. Thus, upon replacing σ with its composite with a suitable $\mathcal{F}$-automorphsim of $N_P(Q)$, we have $\sigma\psi\sigma^{-1} = c_y$ for some $y \in N_P(\sigma N_P(Q))$. Since $\psi|_Q = \varphi$ and $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$, we have $y \in N_P(\sigma(Q))$. In genenral $N_P(\sigma(Q)) \supseteq \sigma N_P(Q)$; since Q is fully $\mathcal{F}$-normalized, we have $N_P(\sigma(Q)) = \sigma N_P(Q)$. So $y \in \sigma N_P(Q)$, so $\psi = c_{\sigma^{-1}(y)}$, which contracts the assumption on ψ. Thus $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ □

PROPOSITION 1.6. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, let $\varphi\colon Q \to P$ be a morphism in $\mathcal{F}$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalized.*

(1) *N_φ is the largest among subgroups U of $N_P(Q)$ containing Q for which there is a morphism $\psi\colon U \to P$ in $\mathcal{F}$ such that $\psi|_Q = \varphi$.*
(2) *There is $\sigma \in \mathrm{Aut}_{\mathcal{F}}(\varphi(Q))$ such that $N_{\sigma\varphi} = N_P(Q)$, i.e. there is a morphism $\psi\colon N_P(Q) \to P$ in $\mathcal{F}$ such that $\psi|_Q = \sigma\varphi$.*

PROOF. (1) Suppose that U is a subgroup of $N_P(Q)$ containing Q and $\psi\colon U \to P$ is a morphism in $\mathcal{F}$. Let $x \in U$. Then, for $u \in \varphi(Q)$,

$$\varphi \circ c_x \circ \varphi^{-1}(u) = \varphi(x\varphi^{-1}(u)x^{-1}) = \psi(x)u\psi(x)^{-1} = c_{\psi(x)}(u);$$

so $x \in N_\varphi$. Therefore $U \subseteq N_\varphi$.

(2) We have that $\varphi\mathrm{Aut}_P(Q)\varphi^{-1}$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(\varphi(Q))$. Since $\mathrm{Aut}_P(\varphi(Q))$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(\varphi(Q))$ by Proposition 1.5, there is $\sigma \in \mathrm{Aut}_{\mathcal{F}}(\varphi(Q))$ such that

$$\sigma\varphi\mathrm{Aut}_P(Q)\varphi^{-1}\sigma^{-1} \subseteq \mathrm{Aut}_P(\varphi(Q)),$$

which means that $N_{\sigma\varphi} = N_P(Q)$. □

PROPOSITION 1.7. *Let $\varphi\colon Q \to P$ be a morphism in $\mathcal{F}$ such that $\varphi(Q)$ is fully $\mathcal{F}$-centralized. Then there exists a morphism $\psi\colon N_\varphi \to P$ such that $\psi|_Q = \varphi$.*

PROOF. Let $\sigma\colon \varphi(Q) \to P$ be a morphism in $\mathcal{F}$ such that $\sigma\varphi(Q)$ is fully $\mathcal{F}$-normalized. By Proposition 1.6, we may assume that $N_{\sigma\varphi} = N_P(Q)$, so there is a morphism $\alpha\colon N_P(Q) \to P$ in $\mathcal{F}$ such that $\alpha|_Q = \sigma\varphi$. By extension axiom, there is also a morphism $\beta\colon N_\sigma \to P$ in $\mathcal{F}$ such that $\beta|_{\varphi(Q)} = \sigma$. We shall show that $\alpha(N_\varphi) \subseteq \beta(N_\sigma)$; then setting $\psi = \beta^{-1} \circ \alpha|_{N_\varphi}$ we get a desired morphism.

Let $y \in N_\varphi$. Then there is $z \in N_P(\varphi(Q))$ such that $\varphi c_y \varphi^{-1} = c_z$ on $\varphi(Q)$. Then

$$\sigma c_z \sigma^{-1} = \sigma\varphi c_y \varphi^{-1}\sigma^{-1} = c_{\alpha(y)}$$

on $\sigma\varphi(Q)$. Thus $z \in N_\sigma$ and $\sigma c_z \sigma^{-1} = c_{\beta(z)}$ on $\sigma\varphi(Q)$. Then $\alpha(y) \in \beta(z)C_P(\sigma\varphi(Q))$. In general $C_P(\sigma\varphi(Q)) \supseteq \sigma C_P(\varphi(Q))$; since $\varphi(Q)$ is fully $\mathcal{F}$-centralized, we have $C_P(\sigma\varphi(Q)) = \sigma C_P(\varphi(Q))$. Thus $\alpha(y) \in \beta(N_\sigma)$. Hence we have $\alpha(N_\varphi) \subseteq \beta(N_\sigma)$, as desired. $\square$

THEOREM 1.8 ([**8**, 1.2]). *Let $\mathcal{F}$ be a category on a finite p-group P. Then $\mathcal{F}$ is a fusion system on P if and only if $\mathcal{F}$ satisfies the following properties:*

(1) $\mathrm{Hom}_P(Q,R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q,R)$ *for all subgroups Q, R of P;*
(2) *if Q is a fully $\mathcal{F}$-normalized subgroup of P, then Q is fully $\mathcal{F}$-centralized and* $\mathrm{Aut}_P(Q)$ *is a Sylow p-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(Q)$*;*
(3) *for every morphism $\varphi\colon Q \to P$ in $\mathcal{F}$ such that $\varphi(Q)$ is fully $\mathcal{F}$-centralized, there is a morphism $\psi\colon N_\varphi \to P$ in $\mathcal{F}$ such that $\psi|_Q = \varphi$.*

PROOF. It follows immediately from Propositions 1.5 and 1.7. $\square$

DEFINITION 1.9. Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Q be a subgroup of P.

(1) Q is *$\mathcal{F}$-centric* if $C_P(Q') \subseteq Q'$ for every $Q' \cong Q$ in $\mathcal{F}$.
(2) Q is *$\mathcal{F}$-radical* if $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)) = 1$.
(3) Q is *$\mathcal{F}$-essential* if Q is $\mathcal{F}$-centric, $Q \neq P$, and $\mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$ has a strongly p-embedded subgroup. A *strongly p-embedded* subgroup of a group G is a proper subgroup H of G which contains a Sylow p-subgroup S of G such that $S \cap {}^xS = 1$ for all $x \in G - H$.

We note some immediate facts:

PROPOSITION 1.10. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Q be a subgroup of P.*

(1) *Q is $\mathcal{F}$-centric if and only if $C_P(Q') = Z(Q')$ for every $Q' \cong Q$ in $\mathcal{F}$.*
(2) *If Q is $\mathcal{F}$-centric, then Q is fully $\mathcal{F}$-centralized.*
(3) *If Q is $\mathcal{F}$-essential, then Q is $\mathcal{F}$-radical.*

PROOF. (1) Clear.

(2) Suppose that Q is $\mathcal{F}$-centric. Then for any morphism $\varphi\colon Q \to P$ in $\mathcal{F}$, we have $\varphi(C_P(Q)) = \varphi(Z(Q)) = Z(\varphi(Q)) = C_P(\varphi(Q))$, so $|C_P(\varphi(Q))| = |C_P(Q)|$. Thus Q is $\mathcal{F}$-centralized.

(3) Suppose that Q is not $\mathcal{F}$-radical, i.e. $A = \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$ has a nontrivial normal p-subgroup R. Then for any Sylow p-subgroup S of A and for any $x \in A$, we have $1 \neq R \subseteq S \cap {}^xS$. Thus Q is not $\mathcal{F}$-essential. $\square$

The following fundamental theorem of fusion systems, which was originally proved by Alperin [**1**] in a slightly weaker form and later extended by Goldschmidt [**18**] and Puig [**30**], says that a fusion system $\mathcal{F}$ on a finite p-group P is completely determined by its automorphism groups of $\mathcal{F}$-essential subgroups of P and P itself.

THEOREM 1.11 (Alperin's fusion theorem). *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Every isomorphism in $\mathcal{F}$ can be written as a composition of finitely many isomorphisms $\varphi\colon Q \to R$ in $\mathcal{F}$ such that $\varphi = \alpha|_Q$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(E)$ where E is either P or an $\mathcal{F}$-essential subgroup of P containing both Q and R.*

For the proof of Alperin's fusion theorem, we refer to [**27**, 5.2].

2. Fusion Systems for Finite Groups

DEFINITION 1.12. Let G be a finite group and P a Sylow p-subgroup of G. We denote by $\mathcal{F}_P(G)$ the category whose objects are subgroups of P and such that

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) = \mathrm{Hom}_G(Q, R)$$

for all subgroups Q, R of P.

PROPOSITION 1.13. *Let G be a finite group and P a Sylow p-subgroup of G.*

(1) *The category $\mathcal{F}_P(G)$ is a fusion system on P.*
(2) *A subgroup Q of P is fully $\mathcal{F}_P(G)$-centralized if and only if $C_P(Q)$ is a Sylow p-subgroup of $C_G(Q)$.*
(3) *A subgroup Q of P is fully $\mathcal{F}_P(G)$-normalized if and only if $N_P(Q)$ is a Sylow p-subgroup of $N_G(Q)$.*

PROOF. Clearly $\mathcal{F}_P(G)$ is a category on P. Let us show (2) and (3) first. Suppose that $C_P(Q)$ is a Sylow p-subgroup of $C_G(Q)$. If ${}^xQ \leq P$, then $C_P({}^xQ)$ is a p-subgroup of $C_G({}^xQ)$. But $|C_G({}^xQ)| = |{}^xC_G(Q)| = |C_G(Q)|$, so $|C_P({}^xQ)| \leq |C_P(Q)|$. Thus Q is fully $\mathcal{F}_P(G)$-centralized. Conversely, suppose that Q is fully $\mathcal{F}_P(G)$-centralized. Let S be a Sylow p-subgroup of $C_G(Q)$ containing $C_P(Q)$. Then QS is a p-subgroup of G, so ${}^x(QS) \leq P$ for some $x \in G$. Then ${}^xQ \leq P$ and ${}^xS \leq C_P({}^xQ)$, so $|C_P(Q)| \leq |S| \leq |C_P({}^xQ)|$. Since Q is fully $\mathcal{F}_P(G)$-centralized, it follows that $C_P(Q) = S$. This proves (2). The same argument with normalizers proves (3).

Now let us prove (1). Clearly $\mathcal{F}_P(P) \subseteq \mathcal{F}_P(G)$. If Q is a fully $\mathcal{F}_P(G)$-normalized subgroup of P, then $N_P(Q)$ is a Sylow p-subgroup of $N_G(Q)$ by (3). Since

$$\mathrm{Aut}_P(Q) \cong N_P(Q)/C_P(Q) \quad \text{and} \quad \mathrm{Aut}_{\mathcal{F}_P(G)}(Q) \cong N_G(Q)/C_G(Q),$$

it follows that $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}_P(G)}(Q)$. It remains to show that the extension axiom holds.

Let $\varphi = c_x \colon Q \to P$ be a morphism in $\mathcal{F}_P(G)$ such that ${}^xQ \le P$ is fully $\mathcal{F}_P(G)$-normalized, that is, $N_P({}^xQ)$ is a Sylow p-subgroup of $N_G({}^xQ)$. If $u \in N_\varphi$ then there exists some $v \in N_P({}^xQ)$ such that $\varphi c_u \varphi^{-1} = c_v$ on ${}^xQ$, or equivalently $xux^{-1} = vw$ for some $w \in C_G({}^xQ)$. Thus ${}^xN_\varphi \le N_P({}^xQ)C_G({}^xQ)$. Since $N_P({}^xQ)C_G({}^xQ)$ is a subgroup of $N_G({}^xQ)$ containing $N_P({}^xQ)$, it has $N_P({}^xQ)$ as a Sylow p-subgroup. Thus there is $c \in C_G({}^xQ)$ such that ${}^{cx}N_\varphi \le N_P({}^xQ) \le P$. Now let $\psi = c_{cx}|_{N_\varphi}$. Then $\psi|_Q = c_{cx}|_Q = c_x|_Q = \varphi|_Q$, proving (1). $\square$

We call $\mathcal{F}_P(G)$ a *fusion system for the finite group G (at the prime p)*. Since Sylow p-subgroups of G are all G-conjugate, fusion systems for G are all equivalent categories.

3. Review of Block Theory

3.1. Idempotents and Blocks of an Algebra. Let A be a finite dimensional (associative unitary) algebra over a field k. An *idempotent* of A is a nonzero element i of A such that $i^2 = i$. Two idempotents i, j of A are said to be *orthogonal* if $ij = ji = 0$. A *decomposition* of an idempotent i of A is a finite set J of pairwise orthogonal idempotents of A such that $i = \sum_{j\in J} j$. An idempotent i of A is called *primitive* if $\{\, i \,\}$ is the only decomposition of i. A decomposition of an idempotent i of A consisting of primitive idempotents is called a *primitive decomposition* of i. If 1_A has a primitive decomposition J in A, we have a decomposition of left A-modules

$$A \cong \bigoplus_{j\in J} Aj$$

where each Aj is a projective indecomposable A-module. Let i, j be two idempotents of A. Krull-Schmidt theorem shows that i, j are *conjugate in A* (i.e. $j = uiu^{-1}$ for some $u \in A^\times$, the multiplicative group of invertible elements of A) if and only if $Ai \cong Aj$ as left A-modules. Since we have an isomorphism of k-vector spaces

$$\begin{array}{ccc} \mathrm{Hom}_A(Ai, Aj) & \cong & iAj \\ f & \mapsto & f(i) \\ (xi \mapsto xic) & \leftarrow\!\shortmid & c, \end{array}$$

we have the following proposition:

PROPOSITION 1.14. *Let A be a finite dimensional algebra over a field k. Two idempotents i, j of A are conjugate in A if and only if there exist $c \in iAj$, $d \in jAi$ such that $cd = i$, $dc = j$.*

An idempotent i of A which lies in the center $Z(A)$ of A is called a *central idempotent* of A. A primitive idempotent i of $Z(A)$ is called a *primitive central idempotent*, or a *block* of A. If 1_A has a primitive decomposition J in $Z(A)$, we have a decomposition of algebras

$$A \cong \prod_{j \in J} Aj$$

where each Aj is an indecomposable algebra. Such Aj is called the *block algebra* of the block j.

Let us observe some simple but useful facts about idempotents.

LEMMA 1.15. *Let A be a finite dimensional algebra over a field k. Let i, j be idempotents of A.*

(1) *j belongs to a decomposition of i if and only if $ij = j = ji$, which in turn holds if and only if $j = iji$.*
(2) *If i is a central idempotent and j is a primitive idempotent in A, then $ij \neq 0$ if and only if $ij = j$.*
(3) *If both i and j are blocks of A, then $ij \neq 0$ if and only if $i = j$.*

PROOF. (1) Let J be a decomposition of i, so that $i = \sum_{j' \in J} j'$. If $j \in J$, then by multiplying j on both sides we get $ij = j = ji$. Conversely, if $ij = j = ji$, then one can easily check that either $i = j$ or $\{ j, i - j \}$ is a decomposition of i. In any case j belongs to a decomposition of i. The second equivalence is obtained by multiplying i on the given identities and using the fact that i is an idempotent.

(2) Suppose that $ij \neq 0$. Since i is central, $\{ ij, (1 - i)j \}$ is a decomposition of j unless $(1 - i)j = 0$. Hence $(1 - i)j = 0$ by the primitivity of j. The converse is obvious.

(3) follows from (2) by changing the role of i and j. $\square$

COROLLARY 1.16. *Let A be a finite dimensional algebra over a field k. Then A has only finitely many blocks.*

PROOF. Since A is finite dimensional, 1_A has a (finite) primitive decomposition J in $Z(A)$. Suppose that i is a block of A. Then we have $i = \sum_{j \in J} ij$, so $ij \neq 0$ for some $j \in J$. By (1.15.3), it follows that $i = j$, proving the assertion. □

Finally we recall two crucial properties of primitive idempotents for later use.

PROPOSITION 1.17 (Rosenberg's Lemma). *Let A be a finite dimensional algebra over a field k. Let i be a primitive idempotent of A. If $i \in \sum_{I \in \Gamma} I$ where Γ is a set of ideals of A, then $i \in I$ for some $I \in \Gamma$.*

PROPOSITION 1.18 (Idempotent Lifting Theorem). *Let A, B be finite dimensional algebras over a field k. Let $f\colon A \to B$ be a surjective k-algebra homomorphism.*

(1) *If i is a primitive idempotent of A, then either $f(i) = 0$ or $f(i)$ is a primitive idempotent of B.*
(2) *If j is a primitive idempotent of B, then there exists a primitive idempotent i of A such that $f(i) = j$.*
(3) *Let i, i' be primitive idempotents of A such that $f(i) \neq 0 \neq f(i')$. Then i and i' are conjugate in A if and only if $f(i)$ and $f(i')$ are conjugate in B.*

Proofs of the above two propositions can be found in [**35**, 3.2, 4.9]. We will need the following slight generalization of Idempotent Lifting Theorem.

PROPOSITION 1.19 (Idempotent Lifting Theorem). *Let A, B be finite dimensional algebras over a field k with ideals I, J, respectively. Let $f\colon A \to B$ be a k-algebra homomorphism such that $f(I) = J$.*

(1) *If i is a primitive idempotent of A contained in I, then either $f(i) = 0$ or $f(i)$ is a primitive idempotent of B.*
(2) *If j is a primitive idempotent of B contained in J, then there exists a primitive idempotent i of A contained in I such that $f(i) = j$.*
(3) *Let i, i' be primitive idempotents of A contained in I such that $f(i) \neq 0 \neq f(i')$. Then i and i' are conjugate in A if and only if $f(i)$ and $f(i')$ are conjugate in B.*

PROOF. (1) Let i be a primitive idempotent of A contained in I, and assume that $f(i) \neq 0$. Applying Proposition 1.18 to $f\colon A \to f(A)$, we have that $f(i)$ is a primitive idempotent in $f(A)$. Suppose that $f(i)$ has a decomposition $\{\, j, j' \,\}$ in A. Then we have $j = jf(i) \in J \subseteq f(A)$; similarly $j' \in f(A)$. Thus $\{\, j, j' \,\}$ is a decomposition of $f(i)$ in $f(A)$, a contradiction. Hence $f(i)$ is a primitive idempotent in A.

(2) Let j is a primitive idempotent of B contained in J. Clearly j is also primitive in $f(A)$. Applying Proposition 1.18 to $f\colon A \to f(A)$, we have that there is a primitive idempotent i of A such that $f(i) = j$. We need to show that $i \in I$. Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A/J(A) \arrow[r, "\overline{f}"] & B/J(B) \end{tikzcd}$$

where $J(A)$, $J(B)$ denote the Jacobson radicals of A, B, respectively, and the vertical arrows are canonical projections. By assumption $\overline{I} = (I + J(A))/J(A)$, $\overline{J} = (J+J(B))/J(B)$ are ideals of $\overline{A} = A/J(A)$, $\overline{B} = B/J(B)$, respectively, and $\overline{f}(\overline{I}) = \overline{J}$. Since $\overline{A}$, $\overline{B}$ are semisimple algebras, so are $\overline{I}$, $\overline{J}$, and hence $\overline{f}\colon \overline{I}/(\overline{I} \cap \mathrm{Ker}(\overline{f})) \to \overline{J}$ is an isomorphism of algebras and $\overline{f}(S) \cap \overline{J} = 0$ for any simple subalgebras S of $\overline{A}$ not contained in $\overline{I}$. It follows that $i + J(A) \in \overline{I}$, i.e. $i \in I + J(A)$. Now $J(A)$ is a nilpotent ideal of A, so $J(A)^n = 0$ for some positive integer n. Since i is an idempotent and I is an ideal of A, we have

$$i = i^n \in (I + J(A))^n \subseteq I + J(A)^n = I.$$

(3) Suppose that $f(i)$ and $f(i')$ are conjugate in B. By Proposition 1.14, there exist $a \in f(i)Bf(i')$, $b \in f(i')Bf(i)$ such that $ab = f(i)$, $ba = f(i')$. Since $f(i)$ is an element of the ideal J of B, we have $a, b \in J \subseteq f(A)$, and hence $a = f(i)af(i') \in f(i)f(A)f(i')$, $b = f(i')bf(i) \in f(i')f(A)f(i)$. Thus $f(i)$ and $f(i')$ are conjugate in $f(A)$. Then applying Proposition 1.18 to $f\colon A \to f(A)$, we have that i, i' are conjugate in A. The opposite direction is obvious. $\square$

From now on, let G be a finite group and let k be a field of prime characteristic p. To analyse the blocks of the group algebra kG, we introduce two instrumental maps on kG, namely the trace map and the Brauer homomorphism, in the subsequent two sections.

3.2. The G-Algebra Structure and the Trace Map on kG. There is a canonical group homomorphism $G \to (kG)^\times$ sending each $x \in G$ to x itself viewed as an element of the group algebra kG. This map induces a natural G-action on kG by conjugation: for $a \in kG$ and $x \in G$, let

$${}^x a := xax^{-1}.$$

In this sense, kG is called a *G-algebra over k*. The notion of G-algebras was first introduced by Green **[19]** and later further developed by Puig **[31]** **[32]**.

For any subgroup P of G, let $(kG)^P$ denote the set of elements of kG fixed by the action of P; that is,

$$(kG)^P = \{\, a \in kG \mid {}^x a = a \text{ for all } x \in P \,\}.$$

Clearly $(kG)^P$ is a k-subalgebra of kG. Moreover, $N_G(P)$-action leaves $(kG)^P$ invariant, hence making it an $N_G(P)$-algebra.

If $Q \subseteq P$ are subgroups of G, then $(kG)^P \subseteq (kG)^Q$, so there is an inclusion map $(kG)^P \hookrightarrow (kG)^Q$. Using the G-algebra structure on kG, we can define a map in the other direction. For $a \in (kG)^Q$, define

$$\mathrm{Tr}_Q^P(a) = \sum_{x \in [P/Q]} {}^x a$$

where $[P/Q]$ denotes a set of representatives of left cosets of Q in P. $\mathrm{Tr}_Q^P(a)$ is well-defined since a is Q-invariant. Moreover $\mathrm{Tr}_Q^P(a) \in (kG)^P$ because for any $y \in P$, $y[P/Q] = \{\, yx \mid x \in [P/Q] \,\}$ is still a set of representatives of left cosets of Q in P. Thus defined k-linear map $\mathrm{Tr}_Q^P \colon (kG)^Q \to (kG)^P$ is called the *trace map* from Q to P on kG.

We summarize some standard properties of the trace map:

PROPOSITION 1.20. *Let P be a subgroup of G.*

(1) *If $R \leq Q \leq P$ and $a \in (kG)^R$, we have $\mathrm{Tr}_Q^P \mathrm{Tr}_R^Q(a) = \mathrm{Tr}_R^P(a)$.*
(2) *If $Q \leq P$, $a \in (kG)^P$, and $b \in (kG)^Q$, we have*

$$a\mathrm{Tr}_Q^P(b) = \mathrm{Tr}_Q^P(ab), \quad \mathrm{Tr}_Q^P(b)a = \mathrm{Tr}_Q^P(ba).$$

In particular, $(kG)_Q^P := \mathrm{Tr}_Q^P((kG)^Q)$ is an ideal of $(kG)^P$.
(3) *(Mackey's formula) If $Q, R \leq P$ and $a \in (kG)^R$, we have*

$$\mathrm{Tr}_R^P(a) = \sum_{x \in [Q\backslash P/R]} \mathrm{Tr}_{Q \cap {}^x R}^Q({}^x a)$$

where $[Q\backslash P/R]$ denotes a set of double coset representatives of Q and R in P.

Finally, we note an easy observation for later use.

PROPOSITION 1.21. *Let $Q \subseteq P$ be subgroups of G.*

(1) *$(kG)^P$ has a k-basis consisting of the P-conjugacy class sums $\mathrm{Tr}_{C_P(x)}^P(x)$, $x \in G$, of G.*

(2) $(kG)^P_Q$ *is spanned by elements of the form* $\mathrm{Tr}^P_{C_Q(x)}(x)$, $x \in G$.

3.3. The Brauer Homomorphism. Let P be a p-subgroup of G, and let

$$\mathrm{Br}^{kG}_P \colon (kG)^P \to kC_G(P)$$

be the truncation map sending $\sum_{x\in G} \lambda_x x \in (kG)^P$ to $\sum_{x\in C_G(P)} \lambda_x x$ where $\lambda_x \in k$. Clearly Br^{kG}_P is a k-linear map. A remarkable fact is that this map is indeed an algebra homomorphism.

PROPOSITION 1.22. Br^{kG}_P *is a split surjective algebra homomorphism with*

$$\mathrm{Ker}(\mathrm{Br}^{kG}_P) = \bigcap_{Q<P} (kG)^P_Q.$$

PROOF. We show that $(kG)^P = kC_G(P) \oplus \sum_{Q<P}(kG)^P_Q$ as k-vector spaces. Since $\sum_{Q<P}(kG)^P_Q$ is an ideal of $(kG)^P$ by (1.20.2), the proposition follows. By (1.21), $(kG)^P$ has a k-basis consisting of the P-conjugacy class sums $\mathrm{Tr}^P_{C_P(x)}(x)$, $x \in G$, of G. But $C_P(x) = P$ iff $x \in C_G(P)$. Thus we have $(kG)^P \subseteq kC_G(P) + \sum_{Q<P}(kG)^P_Q$. Conversely, suppose that $a \in kC_G(P) \cap \sum_{Q<P}(kG)^P_Q$. By (1.21), a is a k-linear combination of elements of the form $\mathrm{Tr}^P_{C_Q(x)}(x)$, $x \in G$. If $yxy^{-1} \in C_G(P)$ for some $y \in P$, then $x \in C_G(P)$, so $\mathrm{Tr}^P_{C_Q(x)}(x) = |P : C_Q(x)|x = 0$. Thus we have $a = 0$, completing the proof. □

In fact, Br^{kG}_P is slightly better than an algebra homomorphism. Both $(kG)^P$ and $kC_G(P)$ are $N_G(P)$-algebras, and Br^{kG}_P preserves the $N_G(P)$-action because the ideal $\bigcap_{Q<P}(kG)^P_Q$ is also $N_G(P)$-invariant; that is, Br^{kG}_P is an $N_G(P)$-algebra homomorphism. The $N_G(P)$-algebra homomorphism Br^{kG}_P is called the *Brauer homomorphism* for P on kG. We write Br_P instead of Br^{kG}_P if it causes no confusion.

The following two propositions analyze the interaction between the Brauer homomorphism and the trace map.

LEMMA 1.23. *Let* P, Q *be* p*-subgroups of* G. *Suppose that* $a \in (kG)^G_P$ *and* $\mathrm{Br}_Q(a) \neq 0$. *Then there exists* $x \in G$ *such that* $Q \subseteq {}^xP$.

PROOF. We have $a = \mathrm{Tr}^G_P(c)$ for some $c \in (kG)^P$. By Mackey's formula (1.20.3),

$$\mathrm{Br}_Q(a) = \mathrm{Br}_Q\mathrm{Tr}^G_P(c) = \sum_{x\in[Q\backslash G/P]} \mathrm{Br}_Q\mathrm{Tr}^Q_{Q\cap {}^xP}({}^xc).$$

Since $\mathrm{Br}_Q(a) \neq 0$, there exists $x \in G$ such that $Q \cap {}^xP = Q$, or $Q \subseteq {}^xP$. □

PROPOSITION 1.24. *Let* P *be a* p*-subgroup of* G. *Then for* $a \in (kG)^P$ *we have*

$$\mathrm{Br}_P\mathrm{Tr}^G_P(a) = \mathrm{Tr}^{N_G(P)}_P\mathrm{Br}_P(a).$$

In particular, $\mathrm{Br}_P((kG)_P^G) = (kC_G(P))_P^{N_G(P)}$.

PROOF. By Mackey's formula (1.20.3), we have

$$\mathrm{Br}_P\mathrm{Tr}_P^G(a) = \sum_{x\in[P\backslash G/P]} \mathrm{Br}_P\mathrm{Tr}_{P\cap {}^xP}^P({}^xa).$$

But $P \cap {}^xP = P$ iff $x \in N_G(P)$. Thus

$$\mathrm{Br}_P\mathrm{Tr}_P^G(a) = \sum_{x\in[N_G(P)/P]} \mathrm{Br}_P({}^xa) = \mathrm{Tr}_P^{N_G(P)}\mathrm{Br}_P(a).$$

□

3.4. Defect Groups of a Block of the Group Algebra kG.

DEFINITION 1.25. Let b be a block of kG. A *defect group* P of the block b is a minimal subgroup of G such that $b \in (kG)_P^G$.

Let P be a defect group of b. Then P is a p-subgroup of G: if S is a Sylow p-subgroup of G, then $|P : S| \neq 0$ in k, and so we have $a = \mathrm{Tr}_S^P(\frac{1}{|P:S|}a)$ for any $a \in (kG)^P$; thus by the transitivity of the trace map (1.20.1), we have $b \in (kG)_S^G$, whence $P = S$ by the minimality of P.

Using Brauer homomorphisms, we can give alternative characterizations of defect groups of a block.

THEOREM 1.26. *Let b be a block of kG. For a p-subgroup P of G, the following conditions are equivalent:*

(1) *P is a defect group of b.*
(2) *P is a maximal subgroup of G such that* $\mathrm{Br}_P(b) \neq 0$.
(3) *We have $b \in (kG)_P^G$ and* $\mathrm{Br}_P(b) \neq 0$.

PROOF. (1) ⇒ (3): Suppose that P is a defect group of b. Then $b = \mathrm{Tr}_P^G(c)$ for some $c \in (kG)^P$. Suppose that $\mathrm{Br}_P(b) = 0$. Then $b = \sum_{Q<P} \mathrm{Tr}_Q^P(b_Q)$ where $b_Q \in (kG)^Q$, and so

$$b = b^2 = b\mathrm{Tr}_P^G(c) = \mathrm{Tr}_P^G(bc) = \sum_{Q<P} \mathrm{Tr}_Q^G(b_Qc) \in \sum_{Q<P}(kG)_Q^G.$$

By Rosenberg's lemma, we have $b \in (kG)_Q^G$ for some $Q < P$, which is a contradiction to the minimality of P. Thus $\mathrm{Br}_P(b) \neq 0$.

(3) $\Rightarrow$ (2): Suppose that $b \in (kG)_P^G$ and $\mathrm{Br}_P(b) \neq 0$. If R is a subgroup of G containing P such that $\mathrm{Br}_R(b) \neq 0$, then by (1.23) we have $R \subseteq {}^xP$ for soem $x \in G$. Comparing orders, we get $P = R$.

(2) $\Rightarrow$ (1): Suppose that P is a maximal subgroup of G such that $\mathrm{Br}_P(b) \neq 0$, and let R be a defect group of b. Again by (1.23), we have $P \subseteq {}^xR$ for some $x \in G$. But ${}^xR$ is also a defect group of b, so we have $\mathrm{Br}_{{}^xR}(b) \neq 0$ by the previous step. By the maximality of P, it follows that $P = {}^xR$. Thus P is a defect group of b. □

Since b is G-invariant, any G-conjugate of P is again a defect group of b. In fact the converse is also true:

PROPOSITION 1.27. *The defect groups of the block b of kG form a single G-conjugacy class of p-subgroups of G.*

PROOF. Let P, Q be two defect groups of the block b. Then by (1.23) and (1.26), we have $Q \subseteq {}^xP$ for some $x \in G$. By changing the role of P and Q, we also have that $P \subseteq {}^yQ$ for some $y \in G$. It follows that $|P| = |Q|$ and so $Q = {}^xP$. □

Finally, we single out a unique block of kG with a particular property. Let $\epsilon\colon kG \to k$ be the *augmentation map* defined by

$$\epsilon\left(\sum_{x\in G} \lambda_x x\right) = \sum_{x\in G} \lambda_x$$

where $\lambda_x \in k$ for $x \in G$. Clearly ϵ is a surjective algebra homomorphism, and moreover $\epsilon(Z(kG)) = k$. Since the algebra k has a unique block 1_k, the Idempotent Lifting Theorem (1.19) tells us that there exists a unique block b_0 of kG such that $\epsilon(b_0) = 1_k$, or equivalently, such that b_0 is not contained in the kernel of the augmentation map, called the *augmentation ideal* of kG. Such a block b_0 is called the *principal block* of kG.

LEMMA 1.28. *Let $Q \subseteq P$ be subgroups of G. Then $\epsilon((kG)_Q^P) \subseteq |P : Q|k$.*

PROOF. By (1.21), $(kG)_Q^P$ is spanned by elements of the form $\mathrm{Tr}_{C_Q(x)}^P(x)$, and

$$\epsilon(\mathrm{Tr}_{C_Q(x)}^P(x)) = |P : C_Q(x)|1_k = |P : Q||Q : C_Q(x)|1_k.$$

□

PROPOSITION 1.29. *Let b_0 be the principal block of kG. Then the defect groups of kG are the Sylow p-subgroups of G.*

PROOF. Suppose that a defect group P of b_0 is properly contained in some Sylow p-subgroup S of G. Then $b_0 \in (kG)_P^G$; but by (1.28), we get

$$\epsilon((kG)_P^G) \subseteq |G : P|k = |G : S||S : P|k = 0,$$

a contradiction. Thus P is a Sylow p-subgroup of G. $\square$

PROPOSITION 1.30. *Let $b \in (kG)^P$ for some p-subgroup P of G. Then we have*

$$\epsilon(b) = \epsilon(\mathrm{Br}_P(b)).$$

PROOF. It follows from (1.28) and that $b - \mathrm{Br}_P(b) \in \sum_{Q<P}(kG)_Q^P$. $\square$

3.5. Brauer's First Main Theorem.

LEMMA 1.31. *Let P be a normal p-subgroup of G. Let $\pi\colon kG \to kG/P$ be the algebra homomorphism induced by the canonical surjective group homomorphism $G \to G/P$. Then we have*

$$\mathrm{Ker}(\pi) = (kG)J(kP).$$

In particular, $\mathrm{Ker}(\pi)$ *is nilpotent, so is contained in $J(kG)$.*

PROOF. Let $a = \sum_{x\in G}\lambda_x x \in kG$ where $\lambda_x \in k$ for $x \in G$. We may write

$$a = \sum_{x\in G/P}\sum_{y\in P}\lambda_{xy}xy = \sum_{x\in G/P} x\left(\sum_{y\in P}\lambda_{xy}y\right).$$

Then $\pi(a) = 0$ iff $\sum_{y\in P}\lambda_{xy} = 0$ for every $x \in G$, that is, iff $\sum_{y\in P}\lambda_{xy}y$ is in the augmentation ideal I of kP. But since P is a p-group and char $k = p$, we have $I = J(kP)$. The lemma follows. $\square$

PROPOSITION 1.32. *Let P be a normal p-subgroup of G. Then*

(1) *Every central idempotent of kG lies in $kC_G(P)$.*
(2) *P is contained in every defect group of every block of kG.*

PROOF. (1) Let e be a central idempotent of kG. Since $e \in (kG)^P$, we may write $e = c + d$ for some $c \in kC_G(P)$ and $d \in \sum_{Q<P}(kG)_Q^P$. Since e is an idempotent and char $k = p$, we have $e = c^{p^n} + d^{p^n}$ for any positive integer n. Hence, to show that $e \in kC_G(P)$, it suffices to show that $\sum_{Q<P}(kG)_Q^P$ is nilpotent.

By (1.31), it suffices to show that for every $Q < P$, $(kG)_Q^P$ is in the kernel of the map π defined in (1.31). By (1.21), $(kG)_Q^P$ is spanned by elements of the form $\mathrm{Tr}_{C_Q(x)}^P(x)$. Then we have

$$\pi(\mathrm{Tr}_{C_Q(x)}^P(x)) = |P : C_Q(x)|\pi(x) = |P : Q||Q : C_Q(x)|\pi(x) = 0,$$

proving the assertion.

(2) Let b be a block of kG. Then $b \in kC_G(P)$, so $\mathrm{Br}_P(b) = b \neq 0$. Thus P is contained in a defect group of b. Since P is normal in G, it follows that P is contained in every defect group of b. $\square$

THEOREM 1.33 (Brauer's First Main Theorem). *Let P be a p-subgroup of G.*

(1) *The Brauer homomorphism Br_P induces a bijection from the set of blocks of kG with defect group P to the set of blocks of $kN_G(P)$ with defect group P.*
(2) *For each block b of kG with defect group P, $\mathrm{Br}_P(b)$ is the $N_G(P)$-orbit sum of a block e of $kC_G(P)$.*
(3) *The image $\overline{e}$ of e in $kC_G(P)/Z(P)$ is a block with trivial defect group.*
(4) *Let $N_G(P, e) = \{\, x \in N_G(P) \mid {}^xe = e \,\}$. Then the* inertial quotient

$$N_G(P, e)/PC_G(P)$$

of e is a p'-group.

PROOF. (1) By (1.26), we have

$$\left\{ \begin{matrix} \text{blocks of } kG \\ \text{with defect group } P \end{matrix} \right\} = \left\{ \begin{matrix} \text{primitive idempotents of } (kG)^G \\ \text{in } (kG)^G_P \text{ but not in } \mathrm{Ker}(\mathrm{Br}_P) \end{matrix} \right\}.$$

Since Br_P is an $N_G(P)$-algebra homomorphism, Br_P maps $(kG)^G$ to $(kC_G(P))^{N_G(P)}$; by (1.24), Br_P maps the ideal $(kG)^G_P$ onto the ideal $(kC_G(P))^{N_G(P)}_P$. Thus, by Idempotent Lifting Theorem (1.19), Br_P induces a bijection

$$\left\{ \begin{matrix} \text{primitive idempotents of } (kG)^G \\ \text{in } (kG)^G_P \text{ but not in } \mathrm{Ker}(\mathrm{Br}_P) \end{matrix} \right\} \to \left\{ \begin{matrix} \text{primitive idempotents of} \\ (kC_G(P))^{N_G(P)} \text{ in } (kC_G(P))^{N_G(P)}_P \end{matrix} \right\}.$$

But by (1.32), we have

$$\left\{ \begin{matrix} \text{blocks of } kN_G(P) \\ \text{with defect group } P \end{matrix} \right\} = \left\{ \begin{matrix} \text{primitive idempotents of} \\ (kC_G(P))^{N_G(P)} \text{ in } (kC_G(P))^{N_G(P)}_P \end{matrix} \right\},$$

proving the assertion.

(2) By the proof of (1), $\mathrm{Br}_P(b) \in (kC_G(P))^{N_G(P)}$. Let e be any block of $kC_G(P)$ such that $\mathrm{Br}_P(b)e = e$. Conjugating both sides by any $x \in N_G(P)$ we get $\mathrm{Br}_P(b){}^xe = {}^xe$. Setting $\widehat{e}$ to be the $N_G(P)$-orbit sum of e, we have $\mathrm{Br}_P(b)\widehat{e} = \widehat{e}$. We have that $\widehat{e}$ is a central idempotent of $kN_G(P)$ by (1.15.3) and (1.20.2). It follows from (1.15.2) that $\mathrm{Br}_P(b) = \widehat{e}$.

(3) Again by the proof of (1), $\mathrm{Br}_P(b) \in (kC_G(P))^{N_G(P)}_P$. In fact, since P is normal in G, one can easily show that $\mathrm{Br}_P(b) \in (kC_G(P))^{PC_G(P)}_P = (kC_G(P))^{C_G(P)}_{Z(P)}$, using

Mackey's formula (1.20). By multiplying e on both sides, we get $e \in (kC_G(P))_{Z(P)}^{C_G(P)}$, and hence $\overline{e} \in (kC_G(P)/Z(P))_1^{C_G(P)/Z(P)}$.

It remains to show that $\overline{e}$ is a block of $kC_G(P)/Z(P)$. First note that the kernel of the surjective algebra homomorphism

$$\pi \colon kC_G(P) \to kC_G(P)/Z(P)$$

induced by the canonical surjective group homomorphism $C_G(P) \to C_G(P)/Z(P)$ is nilpotent by (1.31). Thus e is not in the kernel of π. Since π sends $(kC_G(P))^{C_G(P)}$ to $(kC_G(P)/Z(P))^{C_G(P)/Z(P)}$, and $(kC_G(P))_{Z(P)}^{C_G(P)}$ onto $(kC_G(P)/Z(P))_1^{C_G(P)/Z(P)}$, it follows from Idempotent Lifting Theorem (1.19) that $\overline{e}$ is a block of $kC_G(P)/Z(P)$.

(4) Using the same argument as in the first part of the proof of (3) to $N_G(P,e)$ instead of $PC_G(P)$, we have $\mathrm{Br}_P(b) \in (kC_G(P))_P^{N_G(P,e)}$. By multiplying e on both sides, we get $e = \mathrm{Tr}_P^{N_G(P,e)}(ez) = \mathrm{Tr}_{PC_G(P)}^{N_G(P,e)}\mathrm{Tr}_P^{PC_G(P)}(ez)$ for some $z \in kC_G(P)$. Now $\mathrm{Tr}_P^{PC_G(P)}(ez)$ belongs to the local algebra $Z(kC_G(P)e)$. Thus

$$\mathrm{Tr}_P^{PC_G(P)}(ez) = \lambda e + r$$

for some $\lambda \in k$ and $r \in J(Z(kC_G(P)e))$, and hence

$$e = \mathrm{Tr}_{PC_G(P)}^{N_G(P,e)}(\lambda e + r) = |N_G(P) : PC_G(P)|\lambda e + \mathrm{Tr}_P^{N_G(P,e)}(r).$$

Since $N_G(P,e)$ acts on $Z(kC_G(P)e)$ as algebra automorphisms, $N_G(P,e)$ leaves the Jacobson radical $J(Z(kC_G(P)e))$ invariant. Thus $\mathrm{Tr}_P^{N_G(P,e)}(r) \in J(Z(kC_G(P)e))$. But $e \notin J(Z(kC_G(P)e))$. Therefore $|N_G(P) : PC_G(P)| \neq 0$ in k, that is, $|N_G(P) : PC_G(P)|$ is not divisible by p.

□

3.6. Brauer Pairs.

DEFINITION 1.34. A *Brauer pair* for kG is a pair (P,e) consisting of a p-subgroup P of G and a block e of $kC_G(P)$.

The set of Brauer pairs for kG admits the natural conjugation action by G: for a Brauer pair (P,e) and $x \in G$, let

$${}^x(P,e) := ({}^xP, {}^xe).$$

DEFINITION 1.35. Let (P,e), (Q,f) be Brauer pairs for kG. We say that (P,e) *contains* (Q,f) and write $(P,e) \geq (Q,f)$ if $P \geq Q$ and for every primitive idempotent i of $(kG)^P$ such that $\mathrm{Br}_P(i)e \neq 0$ we have $\mathrm{Br}_Q(i)f = \mathrm{Br}_Q(i)$.

Note that $\mathrm{Br}_P(i)e \neq 0$ if and only if $\mathrm{Br}_P(i)e = \mathrm{Br}_P(i) \neq 0$ because $\mathrm{Br}_P(i)$ is a primitive idempotent of $kC_G(P)$ by (1.15.2) and e is a central idempotent of $kC_G(P)$ (1.19). Also $\mathrm{Br}_P(i) \neq 0$ implies $\mathrm{Br}_Q(i) \neq 0$ by the definition of the Brauer homomorphism. But $\mathrm{Br}_Q(i)f \neq 0$ does not necessarily imply $\mathrm{Br}_Q(i)f = \mathrm{Br}_Q(i)$ because $\mathrm{Br}_Q(i)$ may not be primitive. The relation $\geq$ is compatible with G-conjugation: if $(P,e) \geq (Q,f)$, then ${}^x(P,e) \geq {}^x(Q,f)$ for every $x \in G$. The next theorem is a crucial property of the Brauer pairs. We refer to **[24]** for a concise proof.

THEOREM 1.36. *Let (P,e) be Brauer pairs for G and let Q be a subgroup of P. Then there exists a unique block f of $kC_G(Q)$ such that $(P,e) \geq (Q,f)$. Moreover, if Q is a normal subgroup of P, then such a block f is the unique block of $kC_G(Q)$ which is P-invariant and such that $\mathrm{Br}_P(f)e = e$.*

We write $(P,e) \trianglerighteq (Q,f)$ when $(P,e) \geq (Q,f)$ and $P \trianglerighteq Q$. Using this theorem and the properties of idempotents (1.15), we get the following characterizations of the relation $\geq$ between Brauer pairs.

COROLLARY 1.37. *Let (P,e), (Q,f) be Brauer pairs for G such that $P \geq Q$. Then the following conditions are equivalent:*

(1) $(P,e) \geq (Q,f)$.
(2) *For every primitive idempotent i of $(kG)^P$ such that $\mathrm{Br}_P(i)e \neq 0$, and for every primitive idempotent j of $(kG)^Q$ such that $j = iji$ and $\mathrm{Br}_Q(j) \neq 0$, we have $\mathrm{Br}_Q(j)f \neq 0$.*
(3) *There exists a primitive idempotent i of $(kG)^P$ such that $\mathrm{Br}_P(i)e \neq 0$, and a primitive idempotent j of $(kG)^Q$ such that $j = iji$ and $\mathrm{Br}_Q(j)f \neq 0$.*
(4) *There exists a primitive idempotent i of $(kG)^P$ such that $\mathrm{Br}_P(i)e \neq 0$, $\mathrm{Br}_Q(i)f \neq 0$.*

PROOF. (1) $\Rightarrow$ (2): Suppose (1). Let i be a primitive idempotent of $(kG)^P$ such that $\mathrm{Br}_P(i)e = \mathrm{Br}_P(i) \neq 0$. Let J be a primitive decomposition of i in $(kG)^Q$. Then $J' = \{\, j \in J \mid \mathrm{Br}_Q(j) \neq 0 \,\}$ is a primitive decomposition of $\mathrm{Br}_Q(i)$ in $kC_G(Q)$ by Idempotent Lifting Theorem. Then, for every $j \in J'$, we have

$$\mathrm{Br}_Q(j)f = \mathrm{Br}_Q(j)\mathrm{Br}_Q(i)f = \mathrm{Br}_Q(j)\mathrm{Br}_Q(i) = \mathrm{Br}_Q(j).$$

(2) $\Rightarrow$ (3): Suppose (2). Choose a primitive decomposition I of 1 in $(kG)^P$. Then $I' = \{\, i \in I \mid \mathrm{Br}_P(i) \neq 0 \,\}$ is a primitive decomposition of 1 in $kC_G(P)$; so

$$e = 1 \cdot e = \sum_{i \in I'} \mathrm{Br}_P(i)e.$$

In particular, $\mathrm{Br}_P(i)e \neq 0$ for some $i \in I'$. Then $\mathrm{Br}_P(i) \neq 0$ and so $\mathrm{Br}_Q(i) \neq 0$. Let J be a primitive decomposition of i in $(kG)^Q$. Then $\mathrm{Br}_Q(j) \neq 0$ for some $j \in J$. Then by (2), $\mathrm{Br}_Q(j)f \neq 0$.

(3) $\Rightarrow$ (4): Suppose (3). Then $\mathrm{Br}_Q(i)f\mathrm{Br}_Q(j) = \mathrm{Br}_Q(i)\mathrm{Br}_Q(j) = \mathrm{Br}_Q(j) \neq 0$, and so $\mathrm{Br}_Q(i)f \neq 0$.

(4) $\Rightarrow$ (1): Suppose (4). By Theorem 1.36, there is a unique block f' of $kC_G(Q)$ such that $(P,e) \geq (Q,f')$. Then $\mathrm{Br}_Q(i)f' = \mathrm{Br}_Q(i) \neq 0$. Then $\mathrm{Br}_Q(i)f'f = \mathrm{Br}_Q(i)f \neq 0$, and so $f'f \neq 0$. By Lemma 1.15, it follows that $f = f'$. □

COROLLARY 1.38. *The set of the Brauer pairs for kG together with the relation $\geq$ is a G-poset.*

PROOF. We only need to check the transitivity of the relation $\geq$, which follows from Corollary 1.37. □

DEFINITION 1.39. Let (P,e) be a Brauer pair for kG. (P,e) is called a *b-Brauer pair* if b is the unique block of kG such that $(P,e) \geq (1,b)$.

By (1.36), (P,e) is a b-Brauer pair if and only if $\mathrm{Br}_P(b)e = e$. The next theorem says that b-Brauer pairs satisfy a "Sylow theorem".

PROPOSITION 1.40. *Let b be a block of kG.*

(1) *Let (R,g) be a b-Brauer pair. Then there is a b-Brauer pair (P,e) containing (Q,f) such that P is a defect group of b.*
(2) *Let (P,e), (Q,f) be two b-Brauer pairs such that P, Q are defect groups of b. Then there is $x \in G$ such that $(Q,f) = {}^x(P,e)$.*

PROOF. (1) Since $\mathrm{Br}_R(b)g = g$, we have in particular $\mathrm{Br}_R(b) \neq 0$. Thus R is contained in some defect group P of b.

(2) Since P, Q are defect groups of b, there is $y \in G$ such that $Q = {}^yP$. Then $(Q, {}^ye)$ is also a b-Brauer pair. By Brauer's First Main Theorem, there is $z \in N_G(Q)$ such that $f = {}^{zy}e$. Then $(Q,f) = {}^{zy}(P,e)$. □

4. Fusion Systems for Blocks of Finite Groups

Let k be an algebraically closed field of prime characteristic p, G be a finite group, and let b be a block of kG. Fix a maximal b-Brauer pair (P,e) for kG. For each subgroup Q of P, let e_Q denote the unique block of $kC_G(Q)$ such that $(P,e) \geq (Q,e_Q)$.

DEFINITION 1.41. Let $\mathcal{F}_{(P,e)}(G,b)$ be the category whose objects are subgroups of P and whose morphism sets are given by

$$\mathrm{Hom}_{\mathcal{F}_{(P,e)}(G,b)}(Q,R) = \{\, c_x \in \mathrm{Hom}_G(Q,R) \mid (R,e_R) \geq {}^x(Q,e_Q) \,\}$$

for $Q, R \leq P$.

Note that $(R,e_R) \geq {}^x(Q,e_Q)$ if and only if $R \geq {}^xQ$, ${}^xe_Q = e_{{}^xQ}$ by the uniqueness property of Brauer pairs (1.36).

Clearly $\mathcal{F}_{(P,e)}(G,b)$ is a category on P. Before proving that $\mathcal{F}_{(P,e)}(G,b)$ is a fusion system on P, we characterize fully centralized and fully normalized subgroups of P in $\mathcal{F}_{(P,e)}(G,b)$.

PROPOSITION 1.42. *For $Q \leq P$, the followings are equivalent:*

(1) *Q is fully $\mathcal{F}_{(P,e)}(G,b)$-centralized;*
(2) *$(C_P(Q), e_{QC_P(Q)})$ is a maximal e_Q-Brauer pair for $kC_G(Q)$;*
(3) *$C_P(Q)$ is a defect group of e_Q as a block of $kC_G(Q)$.*

PROOF. $(C_P(Q), e_{QC_P(Q)})$ is indeed a Brauer pair for $kC_G(Q)$ because

$$C_{C_G(Q)}(C_P(Q)) = C_G(QC_P(Q)).$$

Since $QC_P(Q) \trianglerighteq Q$, we have $\mathrm{Br}^{kG}_{QC_P(Q)}(e_Q)e_{QC_P(Q)} = e_{QC_P(Q)}$. But $\mathrm{Br}^{kG}_{QC_P(Q)}(e_Q) = \mathrm{Br}^{kC_G(Q)}_{C_P(Q)}(e_Q)$, so $(C_P(Q), e_{QC_P(Q)})$ is an e_Q-Brauer pair for $C_G(Q)$. In particular, $C_P(Q)$ is contained in a defect group of e_Q.

(1) $\Rightarrow$ (2): Let (S,f) be an e_Q-Brauer pair for $kC_G(Q)$ containing $(C_P(Q), e_{QC_P(Q)})$. Then (QS,f) is a Brauer pair for kG containing $(QC_P(Q), e_{QC_P(Q)})$; in particular (QS,f) is a b-Brauer pair for kG. Since (P,e) is a maximal b-Brauer pair for kG, there exists $x \in G$ such that $(P,e) \geq {}^x(QS,f)$. Then $(P,e) \geq {}^x(Q,e_Q)$ and $C_P({}^xQ) \geq {}^xS$. Since Q is fully $\mathcal{F}_{(P,e)}(G,b)$-centralized, we have $|C_P(Q)| \geq |C_P({}^xQ)|$. But $|C_P({}^xQ)| \geq |{}^xS| = |S| \geq |C_P(Q)|$. Thus we get $S = C_P(Q)$, and hence $(S,f) = (C_P(Q), e_{QC_P(Q)})$.

(2) $\Rightarrow$ (3): Follows from (1.40.1).

(3) $\Rightarrow$ (1): Suppose that $x \in G$ satisfies $(P,e) \geq {}^x(Q,e_Q)$. By the observation at the beginning of the proof, $C_P({}^xQ)$ is contained in a defect group D of $e_{{}^xQ}$ as a block of $kC_G({}^xQ)$. By assumption, ${}^xC_P(Q)$ is a defect group of ${}^xe_Q = e_{{}^xQ}$. Therefore $|C_P(Q)| \geq |C_P({}^xQ)|$, showing that Q is fully $\mathcal{F}_{(P,e)}(G,b)$-centralized. $\square$

The block e_Q may not be a central idempotent in $kN_G(Q)$, but it is a central idempotent in $kN_G(Q, e_Q)$. On the other hand, since Q is a normal p-subgroup of $N_G(Q, e_Q)$, every central idempotent of $kN_G(Q, e_Q)$ is a central idempotent of $kC_G(Q)$ (1.32). Therefore e_Q remains to be a block in $kN_G(Q, e_Q)$. Now applying the same argument in the proof of the previous proposition to normalizers instead of centralizers, we get the following result.

PROPOSITION 1.43. *For $Q \leq P$, the followings are equivalent:*

(1) *Q is fully $\mathcal{F}_{(P,e)}(G, b)$-normalized;*
(2) *$(N_P(Q), e_{N_P(Q)})$ is a maximal e_Q-Brauer pair for $kN_G(Q, e_Q)$;*
(3) *$N_P(Q)$ is a defect group of e_Q as a block of $kN_G(Q, e_Q)$.*

PROPOSITION 1.44. *$\mathcal{F} = \mathcal{F}_{(P,e)}(G, b)$ is a fusion system on P.*

PROOF. For every $Q \leq P$, e_Q is the unique block of $kC_G(Q)$ such that $(P, e) \geq (Q, e_Q)$. Conjugating this containment relation by $x \in P$, we get $(P, e) \geq ({}^xQ, {}^xe_Q)$. It follows from the uniqueness property of Brauer pairs (1.36) that ${}^xe_Q = e_{{}^xQ}$. Thus we have $\mathrm{Hom}_P(Q, R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q, R)$ for every $Q, R \leq P$. Since (P, e) is a maximal b-Brauer pair, P is a defect group of b and $\mathrm{Br}_P(b)e = e$. So by Brauer's first main theorem (1.33), $N_G(P, e)/PC_G(P)$ is a p'-group. Thus $\mathrm{Aut}_P(P) \cong P/Z(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P) \cong N_G(P, e)/C_G(P)$.

It remains to prove the extension axiom. Let Q, R be subgroups of P such that ${}^x(Q, e_Q) = (R, e_R)$ for some $x \in G$ and $(N_P(R), e_{N_P(R)})$ is a maximal e_R-Brauer pair for $kN_G(Q, e_Q)$. Let $\varphi = c_x \colon Q \to R$. Then $N_\varphi = \{\, y \in N_P(Q) \mid \varphi \circ c_y \circ \varphi^{-1} \in \mathrm{Aut}_P(R)\,\}$, so $R \subseteq {}^xN_\varphi \subseteq N_P(R)C_G(R)$. Then $(N_P(R), e_{N_P(R)})$ is a maximal e_R-Brauer pair for $kN_P(R)C_G(R)$, and ${}^x(N_\varphi, e_{N_\varphi})$ is a e_R-Brauer pair for $kN_P(R)C_G(R)$. Thus there exists $c \in C_G(R)$ such that $(N_P(R), e_{N_P(R)}) \geq {}^{cx}(N_\varphi, e_{N_\varphi})$. Then $\psi = c_{cx} \colon N_\varphi \to P$ is a morphism in $\mathcal{F}$ such that $\psi|_Q = \varphi$. $\square$

COROLLARY 1.45. *For $Q \leq P$, the following conditions are equivalent:*

(1) *Q is $\mathcal{F}_{(P,e)}(G, b)$-centric;*
(2) *$Z(Q)$ is a defect group of e_Q as a block of $kC_G(Q)$.*

PROOF. If Q is $\mathcal{F}_{(P,e)}(G, b)$-centric, then Q is fully $\mathcal{F}_{(P,e)}(G, b)$-centralized. Thus $Z(Q) = C_P(Q)$ is a defect group of e_Q as a block of $kC_G(Q)$. Conversely, suppose that $Z(Q)$ is a defect group of e_Q as a block of $kC_G(Q)$. If $(P, e) \geq {}^x(Q, e_Q)$, then $Z({}^xQ)$ is a defect group of $e_{{}^xQ}$ as a block of $kC_G({}^xQ)$, which is contained in $C_P({}^xQ)$. Since $C_P({}^xQ)$ is contained in some defect group of $e_{{}^xQ}$ as a block of $kC_G({}^xQ)$ by the proof of (1.42), it follows that $Z({}^xQ) = C_P({}^xQ)$. Thus Q is $\mathcal{F}_{(P,e)}(G, b)$-centric. $\square$

We call $\mathcal{F}_{(P,e)}(G,b)$ a *fusion system for the block* b. Since all maximal b-Brauer pairs are G-conjugate, fusion systems for b are all equivalent categories.

Fusion systems for finite groups are special cases of fusion systems for blocks of finite groups:

PROPOSITION 1.46. *A fusion system for a finite group G (at the prime p) is a fusion system for the principal block b_0 of kG.*

PROOF. Let b_0 be the principal block of kG with a maximal b_0-Brauer pair (P,e). By (1.29), P is a Sylow p-subgroup of G. By (1.30), the Brauer correspondent $\mathrm{Br}_P(b_0)$ of b_0 is the principal block of $kN_G(P)$. Since $\mathrm{Br}_P(b_0)e = e$, we have $\mathrm{Br}_P(b_0) = \mathrm{Tr}_{N_G(P,e)}^{N_G(P)}(e)$. If $\epsilon(e) = 0$, then $\epsilon({}^x e) = 0$ for every $x \in N_G(P)$, so $\epsilon(\mathrm{Br}_P(b_0)) = 0$, a contradiction. Thus e is the principal block of $kC_G(P)$. Now if $P \trianglerighteq Q$, then $\mathrm{Br}_P(e_Q)e = e$, so by taking ϵ on both sides we get $\epsilon(\mathrm{Br}_P(e_Q)) = 1$. But by (1.30), $\epsilon(\mathrm{Br}_P(e_Q)) = \epsilon(e_Q)$. Thus e_Q is also principal. Continuing this way, we have that e_Q is the principal block of $kC_G(Q)$ for every $Q \leq P$. Since the principal block of a group algebra is its unique block which is not contained in its augmentation ideal, all isomorphisms of group algebras preserve principal blocks. Thus ${}^x e_Q = e_{{}^x Q}$ for every $Q \leq P$ and every $x \in G$. Therefore we have $\mathcal{F}_{(P,e)}(G,b_0) = \mathcal{F}_P(G)$. □

5. Alperin's Weight Conjecture in terms of Fusion Systems

Throughout this section, let k be an algebraically closed field of prime characteristic p.

Let G be a finite group. A *p-local subgroup* of G is the normalizer in G of a nontrivial p-subgroup of G. A main theme of modular representation theory of finite groups is that many representation theoretic invariants of G (and its blocks) are determined by local subgroups of G. For example, Brauer's First Main Theorem (1.33) shows that blocks of kG with nontrivial defect group P are completely determined by the single p-local subgroup $N_G(P)$ of G.

Let b be a block of kG. One of the fundamental invariants of the block b is the number $\ell(b)$ of isomorphism classes of simple kGb-modules. Alperin's weight conjecture predicts precisely how $\ell(b)$ is determined p-locally. For this, we need some terminology.

DEFINITION 1.47. A *weight* of b is a pair (R,w) consisting of a p-subgroup R of G and a block w of $kN_G(R)/R$ with trivial defect group such that $\overline{\mathrm{Br}_R(b)}w = w$, where $\overline{\mathrm{Br}_R(b)}$ denotes the image of $\mathrm{Br}_R(b)$ in $kN_G(R)/R$.

The set of weights of b admits the natural conjugation action by G denoted by ${}^x(R, w) := ({}^xR, {}^xw)$ for $x \in G$.

For a finite dimensional k-algebra A, let $z(A)$ denote the number of blocks of A which are isomorphic to full matrix algebras over k. Then

CONJECTURE 1.48 (Alperin's Weight Conjecture). *$\ell(b)$ is equal to the number of conjugacy classes of weights of b. In other words,*

$$\ell(b) = \sum_R z(k(N_G(R)/R)\overline{\mathrm{Br}_R(b)})$$

where R runs over representatives of conjugacy classes of p-subgroups of G.

Alperin's weight conjecture was first proposed by Alperin **[2]** in 1987. Since then it has been verified for many classes of finite groups, including symmetric groups or finite general linear groups in non-defining characteristics by Alperin-Fong **[4]** and An **[5]**, and p-solvable groups by Isaacs-Navarro **[21]**.

Now we restate Alperin's weight conjecture using fusion systems for blocks. Fix a maximal b-Brauer pair (P, e) for kG, and for each subgroup Q of P, let e_Q denote the unique block of $kC_G(Q)$ such that $(P, e) \geq (Q, e_Q)$. Let $\mathcal{F} = \mathcal{F}_{(P,e)}(G, b)$.

PROPOSITION 1.49. *Alperin's weight conjecture is equivalent to the following identity*

$$\ell(b) = \sum_Q z(k(N_G(Q, e_Q)/Q)\overline{e}_Q)$$

where Q runs over representatives of the $\mathcal{F}$-isomorphism classes of subgroups of P.

This proposition follows from:

LEMMA 1.50. *Let X be a finite group with a normal subgroup N. Let c be a block of kN. Let $H = \{\, x \in X \mid {}^xc = c \,\}$ and let $d = \mathrm{Tr}_H^X(c)$. Then there exists a k-algebra isomorphism*

$$kXd \cong kHc \otimes_k M_n(k)$$

where $n = |X : H|$.

PROOF. Let $[X/H] = \{\, g_i \mid i \in I \,\}$. Then we have a decomposition of k-vector spaces

$$kXd = \bigoplus_{j\in I} kXg_jcg_j^{-1} = \bigoplus_{j\in I} kXcg_j^{-1} = \bigoplus_{i,j\in I} g_ikHcg_j^{-1}.$$

Let $e_{i,j}$ be the $n \times n$ matrix with 1 in the (i, j)-entry and 0 elsewhere. Define a k-linear map

$$\varphi\colon kHc \otimes_k M_n(k) \to kXd$$

by $\varphi(u \otimes e_{i,j}) = g_i u g_j^{-1}$ where $u \in kHc$ and $i, j \in I$. The map φ is a k-linear isomorphism by the above decomposition. Now it remains to show that φ is an algebra homomorphism. Let $x, y \in H$ and $i, j, k, l \in I$. We need to show that

$$g_i x c g_j^{-1} \cdot g_k y c g_l^{-1} = \delta_{j,k} g_i x c y c g_l^{-1}$$

where $\delta_{j,k}$ is the Kronecker delta. It is obviously true when $j = k$. If $j \neq k$, then $z := g_j^{-1} g_k y \notin H$, so $zcz^{-1} \neq c$, so $czcz^{-1} = 0$ by (1.15.3), and hence

$$g_i x c g_j^{-1} \cdot g_k y c g_l^{-1} = g_i x c z c z^{-1} z g_l^{-1} = 0.$$

□

PROOF OF PROPOSITION 1.49. Since $\mathrm{Br}_R(b) \in (kC_G(R))^{N_G(R)}$, we may write $\mathrm{Br}_R(b)$ as a sum of $N_G(R)$-orbit sums of blocks e of $kC_G(R)$ such that $\mathrm{Br}_R(b)e = e$, that is, blocks e of $kC_G(R)$ such that (R, e) is a b-Brauer pair. So we have

$$\ell(b) = \sum_{(R,e)} z(k(N_G(R)/R)\overline{\widehat{e}})$$

where (R, e) runs over representatives of G-orbits of b-Brauer pairs and $\widehat{e}$ denotes the $N_G(R)$-orbit sum of e. Now applying Lemma 1.50 to $X = N_G(R)/R$, $N = C_G(R)/Z(R)$, and $c = \overline{e}$, we get

$$\ell(b) = \sum_{(R,e)} z(k(N_G(R,e)/R)\overline{e}).$$

(Note that $\overline{e}$ is a block of $kC_G(Q)/Z(Q)$; $\overline{\widehat{e}} = \widehat{\overline{e}}$ and the stabilizer of $\overline{e}$ in $N_G(R)/R$ is $N_G(R,e)/R$ by (1.31).) The proposition follows from the fact that the maximal b-Brauer pairs are G-conjugate and the uniqueness property of the e_Q. □

We further reformulate Alperin's weight conjecture. Let $\overline{\mathcal{F}}$ be the category whose objects are subgroups of P and for $Q, R \leq P$,

$$\mathrm{Hom}_{\overline{\mathcal{F}}}(Q, R) = \mathrm{Inn}(R)\backslash\mathrm{Hom}_{\mathcal{F}}(Q, R),$$

where composition of morphisms is induced by composition of morphisms in $\mathcal{F}$. It is easily shown that composition of morphisms in $\overline{\mathcal{F}}$ is well-defined. Let $\overline{\mathcal{F}}^c$ be the full subcategory of $\overline{\mathcal{F}}$ consisting of $\mathcal{F}$-centric subgroups of P.

PROPOSITION 1.51. *For each $\mathcal{F}$-centric subgroup Q of P, there is a canonical class*

$$\alpha(Q) \in H^2(\mathrm{Aut}_{\overline{\mathcal{F}}^c}(Q), k^\times)$$

such that $k_{\alpha(Q)}\mathrm{Aut}_{\overline{\mathcal{F}^c}}(Q)$ is Morita equivalent to $k(N_G(Q,e_Q)/Q)\overline{e}_Q$, and Alperin's weight conjecture is equivalent to the following identity

$$\ell(b) = \sum_Q z(k_{\alpha(Q)}\mathrm{Aut}_{\overline{\mathcal{F}^c}}(Q))$$

where Q runs over representatives of the $\mathcal{F}$-isomorphism classes of $\mathcal{F}$-centric subgroups of P.

This proposition is a consequence of the following two lemmas:

LEMMA 1.52. $z(k(N_G(Q,e_Q)/Q)\overline{e}_Q) = 0$ *unless Q is $\mathcal{F}$-centric.*

PROOF. Suppose $z(k(N_G(Q,e_Q)/Q)\overline{e}_Q) \neq 0$. It means that that there is a block of $kN_G(Q,e_Q)/Q$ with trivial defect group covering the block $\overline{e}_Q$ of $kC_G(Q)/Z(Q)$. Then $\overline{e}_Q$ has trivial defect group. Then e_Q has $Z(Q)$ as a defect group. Thus Q is $\mathcal{F}$-centric by (1.45). □

LEMMA 1.53. *Let X be a finite group with a normal subgroup N. Let c be a G-stable block of kN with trivial defect group. Then there exists a canonical class $\alpha \in H^2(X/N, k^\times)$ such that*

$$kNc \otimes_k k_\alpha X/N \cong kXc$$

PROOF. Since c is G-stable, there exists a group homomorphism

$$X \to \mathrm{Aut}_k(kNc)$$

which sends $x \in X$ to the conjugation by x on kNc. On the other hand, since c is a block of kN with trivial defect group and k is algebraically closed, kNc is a matrix algebra over k. So, by Skolem-Noether theorem, there exists a surjective group homomorphism

$$(kNc)^\times \to \mathrm{Aut}_k(kNc)$$

which sends $u \in (kNc)^\times$ to the conjugation by u on kNc, whose kernel is isomorphic to $k^\times$.

Now, for each $x \in X$, we may choose $i_x \in (kNc)^\times$ such that

$$xux^{-1} = i_x u i_x^{-1}$$

for every $u \in kNc$. Furthermore, we may assume that $i_1 = c$ and

$$i_{xn} = i_x nc$$

for $x \in X$, $n \in N$. Observe that, if $x \in X$, both $x^{-1}i_x$ and $i_x x^{-1}$ centralize every element of kNc. Thus, if $x \in X$, $n \in N$, then

$$i_{nx} = i_{xx^{-1}nx} = i_x x^{-1}nx = ni_x x^{-1}x = ni_x.$$

Let α_0 be the 2-cocycle of X associated with i_x; that is, a function $\alpha_0 \colon X \times X \to k^\times$ such that $i_x i_y = \alpha_0(x,y)i_{xy}$. Then, for $x, y \in X$ and $m, n \in N$, we have

$$\alpha_0(xm, yn) = \alpha_0(x,y)$$

because

$$\begin{aligned} i_{xm}i_{yn} &= \alpha_0(xm,yn)i_{xmyn} = \alpha_0(xm,yn)i_{xyy^{-1}myn} \\ &= \alpha_0(xm,yn)i_{xy}y^{-1}myn = i_x i_y y^{-1}myn \\ &= i_x m i_y y^{-1}yn = i_x m i_y n. \end{aligned}$$

Thus α_0 factors to a 2-cocycle α of X/N.

We claim that the following two maps define a k-algebra isomorphism:

$$\begin{array}{ccc} kNc \otimes_k k_{\alpha(Q)}X/N & \cong & kXc \\ u \otimes xN & \xrightarrow{\varphi} & ui_x^{-1}x \\ i_x \otimes xN & \xleftarrow{\psi} & xc \end{array}$$

where $u \in kNc$, $x \in X$. φ is a well defined k-linear map because if $x \in X$, $n \in N$, then

$$i_{xn}^{-1}xn = (i_x n)^{-1}xn = n^{-1}i_x^{-1}xn = i_x^{-1}x.$$

φ is an algebra homomorphism because if $x, y \in X$, $u, v \in kNc$, then

$$\begin{aligned} &\varphi((u \otimes xN)(v \otimes yN)) = \varphi(\alpha(x,y)uv \otimes xyN) \\ &= \alpha(x,y)uvi_{xy}^{-1}xy = uvi_y^{-1}i_x^{-1}xy \\ &= ui_x^{-1}xvi_y^{-1}y = \varphi(u \otimes xN)\varphi(v \otimes yN). \end{aligned}$$

Let us show that ψ is a well defined k-linear map. Suppose that $xc = x'c$ with $x, x' \in X$. Then $xN = x'N$ because $c \in kN$. Write $x' = xn$ for some $n \in N$. Then $xc = xnc$, so $c = nc$. Thus $i_{x'} = i_x n = i_x nc = i_x c = i_x$. Therefore ψ is well defined. ψ is an algebra homomorphism by a similar argument as for φ. Now clearly

$$\varphi\psi(xc) = \varphi(i_x \otimes xN) = i_x i_x^{-1}x = xc$$

for $x \in X$. Conversely, let $m \in N$, $x \in X$, and write $i_x^{-1} = \sum_{n\in N} \lambda_n nc$. Then

$$\begin{aligned}\psi\varphi(mc \otimes xN) &= \sum_{n\in N} \lambda_n \psi(mnxc) = \sum_{n\in N} \lambda_n i_{mnx} \otimes mnxN \\ &= \sum_{n\in N} \lambda_n mni_x \otimes xN = mci_x^{-1} i_x \otimes xN \\ &= mc \otimes xN.\end{aligned}$$

Thus φ and ψ are k-algebra isomorphisms. □

PROOF OF PROPOSITION 1.51. Let $X = N_G(Q, e_Q)/Q$, $N = C_G(Q)/Z(Q)$, $c = \overline{e}_Q$. Then $N \trianglelefteq X$, and c is a X-stable block of kN with trivial defect group. Moreover,

$$X/N = N_G(Q, e_Q)/QC_G(Q) \cong \mathrm{Aut}_{\overline{\mathcal{F}}^c}(Q).$$

Thus by (1.53), $k(N_G(Q, e_Q)/Q)\overline{e}_Q$ is Morita equivalent to $k_\alpha \mathrm{Aut}_{\overline{\mathcal{F}}^c}(Q)$. Now the proposition follows from (1.49). □

CHAPTER 2

Weighted Fusion Category Algebras

In [**26**], Linckelmann defined weighted fusion category algebras for blocks of finite groups to reformulate Alperin's weight conjecture. This is constructed using twisted category algebra, an analogue of twisted group algebra for categories. It turns out that the weighted fusion category algebra is also quasi-hereditary. We review the notions of twisted category algebra and quasi-hereditary algebra. Then we analyze the Ext-quiver of the weighted fusion category algebra to give an alternative proof of the main theorem of [**26**]. From this proof, we clarify the structure of the weighted fusion category algebra and give some new properties of the weighted fusion category algebras. Finally, we compute the weighted fusion category algebras for tame blocks.

1. Twisted Category Algebras

Let $\mathcal{C}$ be a *finite category*, that is, a category whose object class $\mathrm{Ob}(\mathcal{C})$ is a finite set and whose morphism set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is finite for every $x, y \in \mathrm{Ob}(\mathcal{C})$. Every category appearing in this thesis is a finite category. Let k be a commutative ring with identity. Let $\mathbf{F}(\mathcal{C}, k)$ denote the category whose objects are covariant functors from $\mathcal{C}$ to the category $\mathrm{Mod}(k)$ of left k-modules and whose morphisms are natural transformations between those functors. Denote by $\underline{k}$ the *constant functor at* k in $\mathbf{F}(\mathcal{C}, k)$ which maps every object of $\mathcal{C}$ to the k-module k, and every morphism of $\mathcal{C}$ to the identity map id_k of k. Let $k\mathcal{C}$ be the *category algebra* of $\mathcal{C}$ over k, namely, the k-algebra which is free as a k-module with basis consisting of all the morphisms of $\mathcal{C}$ and such that multiplication is given by

$$\alpha\beta = \begin{cases} \alpha \circ \beta, & \text{if } \alpha \circ \beta \text{ is defined,} \\ 0, & \text{otherwise.} \end{cases}$$

for morphisms α, β, and extended k-linearly.

PROPOSITION 2.1. *Let $\mathcal{C}$ be a finite category and k be a commutative ring with identity. Then there exists an isomorphism of categories*

$$\mathbf{F}(\mathcal{C}, k) \cong \mathrm{Mod}(k\mathcal{C}).$$

PROOF. Let $\Phi\colon \mathbf{F}(\mathcal{C},k) \to \mathrm{Mod}(k\mathcal{C})$ be the functor which sends each covariant functor $M\colon \mathcal{C} \to \mathrm{Mod}(k)$ to $\bigoplus_{x\in \mathrm{Ob}(\mathcal{C})} M(x)$ whose $k\mathcal{C}$-module structure is that if $\alpha\colon x \to y$ is a morphism of $\mathcal{C}$ and $m \in M(z)$ for some $z \in \mathrm{Ob}(\mathcal{C})$, we have

$$\alpha \cdot m = \begin{cases} M(\alpha)(m), & \text{if } x = z,\\ 0, & \text{otherwise;} \end{cases}$$

and which sends each natural transformation $\varphi\colon M \to N$ to the $k\mathcal{C}$-module homorphism

$$\sum_{x\in \mathrm{Ob}\mathcal{C}} \varphi(x)\colon \bigoplus_{x\in \mathrm{Ob}(\mathcal{C})} M(x) \to \bigoplus_{x\in \mathrm{Ob}(\mathcal{C})} N(x).$$

Conversely, let $\Psi\colon \mathrm{Mod}(k\mathcal{C}) \to \mathbf{F}(\mathcal{C},k)$ be the functor which sends each $k\mathcal{C}$-module U to the functor $\mathcal{C} \to \mathrm{Mod}(k)$ which sends each $x \in \mathrm{Ob}(\mathcal{C})$ to the k-module $\mathrm{id}_x U$, and each morphism $\alpha\colon x \to y$ of $\mathcal{C}$ to the k-module homomorphism $\mathrm{id}_x U \to \mathrm{id}_y U$ given by multiplication by α; and which sends each $k\mathcal{C}$-module homomorphism $f\colon U \to V$ to the natural transformation given by $f|_{\mathrm{id}_x U}\colon \mathrm{id}_x U \to \mathrm{id}_x V$ for each $x \in \mathrm{Ob}(\mathcal{C})$. Then we have $\Psi\Phi = \mathrm{id}_{\mathbf{F}(\mathcal{C},k)}$ and $\Phi\Psi = \mathrm{id}_{\mathrm{Mod}(k\mathcal{C})}$; the latter equality follows from that $\{\, \mathrm{id}_x \mid x \in \mathrm{Ob}(\mathcal{C}) \,\}$ is a decomposition of $\mathrm{id}_{k\mathcal{C}}$, and hence for every $k\mathcal{C}$-module U we have a decomposition of k-modules

$$U = \bigoplus_{x\in \mathrm{Ob}(\mathcal{C})} \mathrm{id}_x U.$$

$\square$

In particular, $\mathbf{F}(\mathcal{C},k)$ is an abelian category with enough projectives and injectives. From now on, we identify covariant functors $\mathcal{C} \to \mathrm{Mod}(k)$ with $k\mathcal{C}$-modules via the isomorphism of Proposition 2.1. In particular, the constant functor $\underline{k}$ is identified with

$$\bigoplus_{x\in \mathrm{Ob}(\mathcal{C})} k_x$$

where $k_x \cong k$ for each $x \in \mathrm{Ob}(\mathcal{C})$. Let 1_x denote the identity element of k_x for each $x \in \mathrm{Ob}(\mathcal{C})$.

DEFINITION 2.2. Let M be a functor in $\mathbf{F}(\mathcal{C},\mathbb{Z})$ and n a nonnegative integer. The *degree n cohomology of the category $\mathcal{C}$ over M* is

$$H^n(\mathcal{C};M) = \mathrm{Ext}^n_{\mathbf{F}(\mathcal{C},\mathbb{Z})}(\underline{\mathbb{Z}}, M).$$

We want to have an explicit description of cocycles and coboundaries for cohomologies of categories. As in the group cohomology, there is a *standard resolution* $\mathcal{P}_*$ of $\underline{\mathbb{Z}}$. For $n \geq 1$, let $\mathcal{C}^n_\sharp$ be the set of n-tuples of composable morphisms of $\mathcal{C}$, that is,

$$\mathcal{C}_\sharp^n = \{\, (\varphi_1, \ldots, \varphi_n) \mid \varphi_i \text{ morphisms of } \mathcal{C} \text{ s.t. } \varphi_1 \circ \cdots \circ \varphi_n \text{ is defined} \,\}.$$

Let

$$\mathcal{P}_n = \bigoplus_{(\varphi_0,\ldots,\varphi_n)\in\mathcal{C}_\sharp^{n+1}} \mathbb{Z}(\varphi_0, \ldots, \varphi_n) \qquad (n \geq 0)$$

with the $\mathbb{Z}\mathcal{C}$-module structure given by composition with the first component: if φ is a morphism of $\mathcal{C}$ and $(\varphi_0, \varphi_1, \ldots, \varphi_n) \in \mathcal{C}_\sharp^{n+1}$, then

$$\varphi \cdot (\varphi_0, \varphi_1, \ldots, \varphi_n) = \begin{cases} (\varphi \circ \varphi_0, \varphi_1, \ldots, \varphi_n), & \text{if } \varphi \circ \varphi_0 \text{ is defined;} \\ 0, & \text{otherwise.} \end{cases}$$

Define the $\mathbb{Z}$-linear map $\partial_n \colon \mathcal{P}_n \to \mathcal{P}_{n-1}$ for $n > 0$ by

$$\partial_n(\varphi_0, \ldots, \varphi_n) = \sum_{i=0}^{n-1}(-1)^i(\varphi_0, \ldots, \varphi_i \circ \varphi_{i+1}, \ldots, \varphi_n) + (-1)^n(\varphi_0, \ldots, \varphi_{n-1}),$$

and $\partial_0 \colon \mathcal{P}_0 \to \underline{\mathbb{Z}}$ by

$$\partial_0(\varphi) = 1_y$$

where $\varphi \colon x \to y$ is a morphism of $\mathcal{C}$. Clearly ∂_n is a $\mathbb{Z}\mathcal{C}$-module homomorphism for all $n \geq 0$.

PROPOSITION 2.3. *$\mathcal{P}_* = \{\, (\mathcal{P}_n, \partial_n) \,\}_{n\geq 0}$ is a projective resolution of $\underline{\mathbb{Z}}$ in $\mathbf{F}(\mathcal{C}, \mathbb{Z})$.*

PROOF. Each $\mathcal{P}_n$ is a projective $\mathbb{Z}\mathcal{C}$-module because

$$\begin{aligned} \mathcal{P}_n &= \bigoplus_{(\varphi_0,\varphi_1,\ldots,\varphi_n)\in\mathcal{C}_\sharp^{n+1}} \mathbb{Z}(\varphi_0, \varphi_1, \ldots, \varphi_n) \\ &= \bigoplus_{\substack{x\in \mathrm{Ob}(\mathcal{C}) \\ (\mathrm{id}_x,\varphi_1,\ldots,\varphi_n)\in\mathcal{C}_\sharp^{n+1}}} \mathbb{Z}\mathcal{C}\mathrm{id}_x(\mathrm{id}_x, \varphi_1, \ldots, \varphi_n). \end{aligned}$$

To show that $\mathcal{P}_*$ is a resolution of $\underline{\mathbb{Z}}$, it suffices to find $\mathbb{Z}$-linear maps $h_n \colon \mathcal{P}_n \to \mathcal{P}_{n+1}$ such that $\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n = \mathrm{id}_{\mathcal{P}_n}$ for all integer n (set $\mathcal{P}_{-1} = \underline{\mathbb{Z}}$, $\mathcal{P}_n = 0$ for $n < -1$, and $\partial_n = 0$ for $n < 0$). One immediately checks that $h_n \colon \mathcal{P}_n \to \mathcal{P}_{n+1}$ given by

$$h_n(\varphi_0, \ldots, \varphi_n) = (\mathrm{id}_x, \varphi_0, \ldots, \varphi_n)$$

where $(\varphi_0, \ldots, \varphi_n) \in \mathcal{C}_\sharp^{n+1}$ and $(\mathrm{id}_x, \varphi_0, \ldots, \varphi_n) \in \mathcal{C}_\sharp^{n+2}$ for $n \geq 0$, $h_{-1}(1_x) = \mathrm{id}_x$ for $x \in \mathrm{Ob}(\mathcal{C})$, and $h_n = 0$ for $n < -1$ satisfies the required property. $\square$

For the convenience in the later use, we modify the standard resolution a little bit and define the *normalized standard resolution* $\overline{\mathcal{P}}_*$ of $\underline{\mathbb{Z}}$. First, let

$$\mathcal{T}_n = \bigoplus_{\substack{x\in\mathrm{Ob}(\mathcal{C})\\ (\mathrm{id}_x,\varphi_1,\ldots,\varphi_n)\in\mathcal{C}_\sharp^{n+1}\\ \varphi_i=\mathrm{id}\text{ for some } i}} \mathbb{Z}\mathcal{C}\mathrm{id}_x(\mathrm{id}_x,\varphi_1,\ldots,\varphi_n)$$

for $n > 0$; let $\mathcal{T}_0 = 0$. Clearly every $\mathcal{T}_n$ is a direct summand of $\mathcal{P}_n$ as $\mathbb{Z}\mathcal{C}$-module with complement

$$\mathcal{T}_n' = \bigoplus_{\substack{x\in\mathrm{Ob}(\mathcal{C})\\ (\mathrm{id}_x,\varphi_1,\ldots,\varphi_n)\in\mathcal{C}_\sharp^{n+1}\\ \varphi_i\neq\mathrm{id}\text{ for all } i}} \mathbb{Z}\mathcal{C}\mathrm{id}_x(\mathrm{id}_x,\varphi_1,\ldots,\varphi_n),$$

and $\partial_n(\mathcal{T}_n) \subseteq \mathcal{T}_{n-1}$, $h_n(\mathcal{T}_n) \subseteq \mathcal{T}_{n+1}$ for every n. It follows that

$$\overline{\mathcal{P}}_n = \mathcal{P}_n/\mathcal{T}_n \cong \mathcal{T}_n'$$

with the induced boundary map $\overline{\partial}_n \colon \overline{\mathcal{P}}_n \to \overline{\mathcal{P}}_{n-1}$ makes a projective resolution of $\underline{\mathbb{Z}}$ in $\mathbf{F}(\mathcal{C},\mathbb{Z})$.

Now if M is a functor in $\mathbf{F}(\mathcal{C},\mathbb{Z})$, then

$$\begin{aligned}\mathrm{Hom}_{\mathbf{F}(\mathcal{C},\mathbb{Z})}(\mathcal{P}_n, M) &\cong \bigoplus_{\substack{x\in\mathrm{Ob}(\mathcal{C})\\ (\mathrm{id}_x,\varphi_1,\ldots,\varphi_n)\in\mathcal{C}_\sharp^{n+1}}} \mathrm{Hom}_{\mathbf{F}(\mathcal{C},\mathbb{Z})}(\mathbb{Z}\mathcal{C}\mathrm{id}_x, M)\\ &\cong \bigoplus_{\substack{x\in\mathrm{Ob}(\mathcal{C})\\ (\mathrm{id}_x,\varphi_1,\ldots,\varphi_n)\in\mathcal{C}_\sharp^{n+1}}} \mathrm{id}_x M\end{aligned}$$

as $\mathbb{Z}$-modules. In particular, if $M = \underline{k}^\times$, the constant functor at the multiplicative group of invertible elements $k^\times$ of k, we get a $\mathbb{Z}$-module isomorphism

$$\mathrm{Hom}_{\mathbf{F}(\mathcal{C},\mathbb{Z})}(\mathcal{P}_n, \underline{k}^\times) \cong \bigoplus_{(\varphi_1,\ldots,\varphi_n)\in\mathcal{C}_\sharp^n} k^\times \cong \mathrm{Map}(\mathcal{C}_\sharp^n, k^\times)$$

where $\mathrm{Map}(\mathcal{C}_\sharp^n, k^\times)$ denotes the set of all functions from $\mathcal{C}_\sharp^n$ to $k^\times$ with the $\mathbb{Z}$-module structure given by pointwise multiplication of functions. Furthermore, the induced coboundary map $\delta^n = \partial_{n+1}^* \colon \mathrm{Hom}_{\mathbf{F}(\mathcal{C},\mathbb{Z})}(\mathcal{P}_n, \underline{k}^\times) \to \mathrm{Hom}_{\mathbf{F}(\mathcal{C},\mathbb{Z})}(\mathcal{P}_{n+1}, \underline{k})$ is given by

$$\begin{gathered}\delta^n(\alpha)(\varphi_1,\ldots,\varphi_{n+1}) = \alpha(\varphi_2,\ldots,\varphi_{n+1}) \prod_{i=1}^{n} \alpha(\varphi_1,\ldots,\varphi_i\circ\varphi_{i+1},\ldots,\varphi_{n+1})^{(-1)^i}\\ \alpha(\varphi_1,\ldots,\varphi_n)^{(-1)^{n+1}}.\end{gathered}$$

Note that

$$\mathrm{Hom}_{\mathbf{F}(\mathcal{C},\mathbb{Z})}(\overline{\mathcal{P}}_n, \underline{k}^\times) \cong \mathrm{Map}_1(\mathcal{C}_\sharp^n, k^\times)$$

where

$$\mathrm{Map}_1(\mathcal{C}_\sharp^n, k^\times) = \{\, \alpha \in \mathrm{Map}(\mathcal{C}_\sharp^n, k^\times) \mid \alpha(\varphi_1, \ldots, \varphi_n) = 1 \text{ if } \varphi_i = \mathrm{id} \text{ for some } i \,\}.$$

Thus, in particulr, a (normalized) 2-cocycle of $\mathcal{C}$ over $k^\times$ is a map $\alpha\colon \mathcal{C}_\sharp^2 \to k^\times$ such that

$$\alpha(\eta \circ \psi, \varphi)\alpha(\eta, \psi) = \alpha(\eta, \psi \circ \varphi)\alpha(\psi, \varphi) \quad \text{for every } (\eta, \psi, \varphi) \in \mathcal{C}_\sharp^3,$$
$$\alpha(1, \varphi) = 1 = \alpha(\varphi, 1) \quad \text{for every } \varphi \in \mathcal{C}_\sharp^1;$$

a 2-coboundary of $\mathcal{C}$ over $k^\times$ is a map $\alpha\colon \mathcal{C}_\sharp^2 \to k^\times$ such that there exists a map $\beta\colon \mathcal{C}_\sharp^1 \to k^\times$ which satisfies

$$\alpha(\psi, \varphi) = \beta(\psi)\beta(\varphi)\beta(\psi \circ \varphi)^{-1}$$

for every $(\psi, \varphi) \in \mathcal{C}_\sharp^2$.

DEFINITION 2.4. Let $\mathcal{C}$ be a finite category, k a commutative ring with identity, and α a 2-cocycle of $\mathcal{C}$ over $k^\times$. Then the *twisted category algebra of $\mathcal{C}$ by α over k*, denoted by $k_\alpha\mathcal{C}$, is the k-algebra which is free as a k-module with basis consisting of morphisms of $\mathcal{C}$ and multiplication is given by

$$\psi\varphi = \begin{cases} \alpha(\psi, \varphi)\psi \circ \varphi, & \text{if } \psi \circ \varphi \text{ is defined;} \\ 0, & \text{otherwise} \end{cases}$$

where ψ, φ are morphisms of $\mathcal{C}$.

The twisted category algebra $k_\alpha\mathcal{C}$ is indeed an algebra: the 2-cocycle condition of α translates exactly to the associativity of multiplication, and the identity element is $\sum_{x \in \mathrm{Ob}(\mathcal{C})} \mathrm{id}_x$. Moreover, if two 2-cocycles α, α' represent the same cohomology class in $H^2(\mathcal{C}, \underline{k}^\times)$, then we have a k-algebra isomorphism

$$k_\alpha\mathcal{C} \cong k_{\alpha'}\mathcal{C}.$$

Indeed, if $\alpha(\psi, \varphi) = \alpha'(\psi, \varphi)\beta(\psi)\beta(\varphi)\beta(\psi \circ \varphi)^{-1}$ for every $(\psi, \varphi) \in \mathcal{C}_\sharp^2$ where $\beta\colon \mathcal{C}_\sharp^1 \to k^\times$, then the map defined by

$$\varphi \mapsto \beta(\varphi)\varphi$$

gives a k-algebra isomorphism from $k_\alpha\mathcal{C}$ to $k_{\alpha'}\mathcal{C}$.

2. Quasi-hereditary Algebras

Quasi-hereditary algebras were first defined by Cline, Parshall and Scott [**9**]. We review the definition and some basic properties of quasi-hereditary algebras following [**13**].

Let A be a finite dimensional algebra over an algebraically closed field k. Let Λ^+ be a finite indexing set such that $\{ L_\lambda \mid \lambda \in \Lambda^+ \}$ is a set of representatives of isomorphism classes of simple A-modules. For each $\lambda \in \Lambda^+$, let $P(\lambda)$ and $I(\lambda)$ be the projective cover and the injective hull of $L(\lambda)$, respectively. Note that $M(\lambda) := J(A)P(\lambda)$ is the unique maximal submodule of $P(\lambda)$. For a finitely generated A-module V and $\lambda \in \Lambda^+$, write $[V, L(\lambda)]$ for the multiplicity of $L(\lambda)$ as a composition factor of V.

For a finitely generated A-module V and a subset π of Λ^+, we say that V *belongs to* π if and only if every composition factor of V is of the form $L(\lambda)$ for some $\lambda \in \pi$. Let $O_\pi(V)$ be the largest submodule of V which belongs to π, and $O^\pi(V)$ the smallest submodule W of V such that V/W belongs to π.

Now fix a partial ordering $\leq$ on Λ^+. For each $\lambda \in \Lambda^+$, let $\pi(\lambda) = \{ \mu \in \Lambda^+ \mid \mu < \lambda \}$. Define $K(\lambda) = O^{\pi(\lambda)}(M(\lambda))$ and $\Delta(\lambda) = P(\lambda)/K(\lambda)$. $\Delta(\lambda)$ is called the *standard module* for $\lambda \in \Lambda^+$. The *costandard module* $\nabla(\lambda)$ is defined by $\nabla(\lambda)/L(\lambda) = O_{\pi(\lambda)}(I(\lambda)/L(\lambda))$.

Now we give a definition of the quasi-hereditary algebra.

DEFINITION 2.5. The algebra A is called *quasi-hereditary* (with respect to the partial ordering $\leq$ on Λ^+) if and only if $I(\lambda)/\nabla(\lambda)$ has a filtration with factors of the form

$$\nabla(\mu), \quad \mu > \lambda$$

for every $\lambda \in \Lambda^+$.

This definition has a dual version.

PROPOSITION 2.6 ([**13**, A3.5]). *The algebra A is quasi-hereditary if and only if $K(\lambda)$ has a filtration with factors of the form*

$$\Delta(\mu), \quad \mu > \lambda$$

for every $\lambda \in \Lambda^+$.

Now suppose that A is a quasi-hereditary algebra with respect to the partial ordering $\leq$ on Λ^+. A finitely generated A-module V is said to have Δ-*filtration* if there is a sequence

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

of submodules of V such that $V_i/V_{i-1} \cong \Delta(\lambda_i)$ for some $\lambda_i \in \Lambda^+$, $1 \leq i \leq n$. Modules with ∇-*filtration* are defined similarly. If an A-module V has both Δ-filtration and ∇-filtration, we say that V is called a *tilting module*.

PROPOSITION 2.7 ([**13**, A4.2]). *Suppose that A is a quasi-hereditary algebra with respect to the partial ordering $\leq$ on Λ^+.*

(1) *For every $\lambda \in \Lambda^+$, there is a unique (up to isomorphism) indecomposable tilting module $T(\lambda)$ such that $T(\lambda)$ has $L(\lambda)$ as a composition factor with multiplicity 1 and all other composition factors of $T(\lambda)$ are of the form $L(\mu)$, $\mu < \lambda$.*
(2) *Every tilting module is a direct sum of the modules $T(\lambda)$, $\lambda \in \Lambda^+$*

Let $T = \oplus_{\lambda \in \Lambda^+} T(\lambda)$. The endomorphism algebra $A' = \mathrm{End}_A(T)^{\mathrm{op}}$ is called the *Ringel dual* of the algebra A.

PROPOSITION 2.8 ([**13**, A4.7]). *Suppose that A is a quasi-hereditary algebra with respect to the partial ordering $\leq$ on Λ^+. Then its Ringel dual A' is also a quasi-hereditary algebra with respect to the opposite order $\leq'$ to $\leq$ on the poset Λ^+.*

We note an important property of a quasi-hereditary algebra.

DEFINITION 2.9. Let R be a ring. R is said to have *finite global dimension* if there is an integer n such that $\mathrm{Ext}^i_R(U, V) = 0$ for all R-modules U, V and all integers $i > n$. In this case, the smallest of such integers n is called the *global dimension* of R.

DEFINITION 2.10. For $\lambda \in \Lambda^+$, let $l(\lambda)$ be the length l of the longest chain $\lambda_0 < \lambda_1 < \cdots < \lambda_l = \lambda$ in Λ^+. Let $l(\Lambda^+) = \max\{\, l(\lambda) \mid \lambda \in \lambda^+ \,\}$.

PROPOSITION 2.11 ([**13**, A2.3]). *Suppose that A is a quasi-hereditary algebra with respect to the partial ordering $\leq$ on Λ^+. Then A has global dimension $\leq 2l(\Lambda^+)$.*

3. The Quivers and Relations of Algebras

3.1. Morita Theory and Basic Algebras.

DEFINITION 2.12. We say that two rings A and B are *Morita equivalent* if their module categories $\mathrm{Mod}(A)$ and $\mathrm{Mod}(B)$ are equivalent as abelian categories.

PROPOSITION 2.13. *Let A be an algebra over a field k. If e is an idempotent of A such that $A = AeA$, then A and eAe are Morita equivalent.*

PROOF. We show that the functor

$$eA \otimes_A -\colon \mathrm{Mod}(A) \to \mathrm{Mod}(eAe)$$

has as inverse the functor

$$Ae \otimes_{eAe} -\colon \mathrm{Mod}(eAe) \to \mathrm{Mod}(A).$$

Clearly, $eA \otimes_A Ae \cong eAe$ and $eAe \otimes_{eAe} -$ is the identity functor on $\mathrm{Mod}(eAe)$. Conversely, the map

$$\begin{array}{rcl} Ae \otimes_{eAe} eA & \to & A \\ ce \otimes ed & \mapsto & ced \end{array}$$

is an isomorphism of A-A-bimodules. It is surjective by the assumption $A = AeA$. Let $\sum_{i \in I} c_i \otimes d_i$ be in the kernel of this map, that is, $\sum_{i \in I} c_i d_i = 0$. Since $A = AeA$, there are $x_j \in Ae$, $y_j \in eA$ such that $\sum_{j \in J} x_j y_j = 1$. Then

$$0 = \sum_{i \in I, j \in J} x_j \otimes y_j c_i d_i = \sum_{i \in I, j \in J} x_j y_j c_i \otimes d_i = \sum_{i \in I} c_i \otimes d_i$$

because $y_j c_i \in eAe$. Thus the map is also injective. □

Let A be a finite dimensional algebra over an algebraically closed field k. Choose a set J of representatives of conjugacy classes of primitive idempotents of A so that elements of J are pairwise orthogonal. Then $\{ Aj \mid j \in J \}$ is a set of representatives of isomorphism classes of projective indecomposable A-modules, and setting

$$S_j = Aj/J(A)j,$$

$\{ S_j \mid j \in J \}$ is a set of representatives of isomorphism classes of simple A-modules.

Let $e = \sum_{j \in J} j$. Then e is an idempotent of A and $A = AeA$. Therefore A is Morita equivalent to eAe by (2.13).

We have an isomorphism of k-vector spaces

$$eAe \cong \bigoplus_{i,j \in J} jAi.$$

We also have

$$jAi \cong \mathrm{Hom}_A(Aj, Ai).$$

Since Ai is a projective A-module, the canonical k-linear map

$$\mathrm{Hom}_A(Aj, Ai) \to \mathrm{Hom}_A(S_j, S_i)$$

is a surjection with kernel $\mathrm{Hom}_A(Aj, J(A)i)$. So we have a short exact sequence of k-vector spaces

$$0 \to \mathrm{Hom}_A(Aj, J(A)i) \to \mathrm{Hom}_A(Aj, Ai) \to \mathrm{Hom}_A(S_j, S_i) \to 0.$$

But since k is algebraically closed, we have

$$\mathrm{Hom}_A(S_j, S_i) \cong \begin{cases} k, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$jAi = \begin{cases} ki \oplus iJ(A)i, & \text{if } i = j; \\ jJ(A)i, & \text{otherwise.} \end{cases}$$

It follows that $J(eAe) = eJ(A)e$ and

$$eAe/J(eAe) \cong \prod_{i \in J} k\bar{i}$$

as k-aglebras, where $\bar{i}$ denotes the image of i in $eAe/J(eAe)$. That is, eAe is a k-algebra which is Morita equivalent to A and every simple eAe-module is one dimensional. In this sense, eAe is called the *basic algebra* of A.

3.2. Quivers with Relations. Quivers with relations are convenient tools to describe the structure of a finite dimensional algebra over an algebraically closed field. We review basic theory of quivers with relations following **[6]**.

DEFINITION 2.14. A *quiver* Q is an oriented graph consisting of a set of *vertices* and a set of *arrows* between vertices, possibly with multiple arrows and loops. The initial vertex of an arrow α is called the *source* of α and denoted by $s(\alpha)$; the terminal vertex of α is called the *target* of α and denoted by $t(\alpha)$.

A *path of length* $n(\geq 1)$ from a vertex x to another vertex y of a quiver Q is a sequence $(\alpha_n, \alpha_{n-1}, \dots, \alpha_1)$ of arrows of Q such that $s(\alpha_1) = x$, $t(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, 2, \dots, n-1$, $t(\alpha_n) = y$. To each vertex x, we assign a unique symbol e_x and call it the *path of length 0 from x to x*, or simply the *path of length 0 at x*.

The *path algebra* kQ of a quiver Q over a field k is the k-algebra whose k-basis consists of all the paths of Q and multiplication is given on basis elements by concatenation of paths:

$$(\beta_m, \dots, \beta_1) \cdot (\alpha_n, \dots, \alpha_1) = \begin{cases} (\beta_m, \dots, \beta_1, \alpha_n, \dots, \alpha_1), & \text{if } t(\alpha_n) = s(\beta_1); \\ 0, & \text{otherwise.} \end{cases}$$

$$(\alpha_n, \dots, \alpha_1) \cdot e_x = \begin{cases} (\alpha_n, \dots, \alpha_1), & \text{if } s(\alpha_1) = x; \\ 0, & \text{otherwise.} \end{cases}$$

$$e_x \cdot (\alpha_n, \dots, \alpha_1) = \begin{cases} (\alpha_n, \dots, \alpha_1), & \text{if } t(\alpha_n) = x; \\ 0, & \text{otherwise.} \end{cases}$$

$$e_x \cdot e_y = \begin{cases} e_x, & \text{if } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

In fact, we can view a quiver Q as a small category whose object set is the set of vertices Q and whose morphism set from x to y is the set of paths of Q from x to y, and such that composition of morphisms is given by concatenation of paths. Note that for each object x, e_x is the identity morphism of x. Then the path algebra kQ is exactly the category algebra of Q over k.

We note some immediate facts:

PROPOSITION 2.15. *Let Q be a quiver and let kQ be the path algebra of Q over a field k.*

(1) *kQ is finitely generated as a k-algebra if and only if Q has finitely many vertices and arrows.*
(2) *kQ is finite dimensional as a k-vector space if and only if Q has finitely many vertices and arrows and no oriented cycles.*
(3) *If the vertex set of Q is finite, then $\{ e_x \mid x \text{ is a vertex of } Q \}$ is a decomposition of 1_{kQ}.*

DEFINITION 2.16. Let Q be a quiver and let kQ be the path algebra of Q over a field k. For each $n \geq 0$, let $kQ_{(n)}$ be the k-linear subspace of kQ with basis consisting of paths of Q of length $\geq n$. An ideal I of kQ contained in $kQ_{(2)}$ is called an *admissible ideal* of kQ.

Note that $kQ_{(n)}$ is an ideal of kQ.

DEFINITION 2.17. A pair (Q, I) consisting of a quiver Q and an admissible ideal I of the path algebra kQ of Q over a field k is called a *quiver with relations*. The quotient algebra kQ/I is called the *path algebra* of the quiver with relations (Q, I).

3.3. The Ext-quiver of an Algebra. Let A be a finite dimensional algebra over an algebraically closed field k. We use the same notation as in 3.1.

DEFINITION 2.18. The *Ext-quiver* $Q(A)$ of A is the quiver whose vertex set is J and such that for each pair $i, j \in J$, the number of arrows from i to j is equal to

$$n_{ji} = \dim_k \mathrm{Ext}^1_A(S_i, S_j).$$

PROPOSITION 2.19. *We have*

$$\begin{aligned} n_{ji} &= \dim_k \mathrm{Hom}_A(S_j, J(A)i/J(A)^2 i) \\ &= \dim_k jJ(A)i/jJ(A)^2 i. \end{aligned}$$

PROOF. From the short exact sequence $0 \to J(A)i \to Ai \to S_i \to 0$ of A-modules, we get an exact sequence of k-vector spaces

$$\mathrm{Hom}_A(Ai, S_j) \to \mathrm{Hom}_A(J(A)i, S_j) \to \mathrm{Ext}^1_A(S_i, S_j) \to \mathrm{Ext}^1_A(Ai, S_j).$$

The first map is a zero map because $J(A)S_j = 0$; $\mathrm{Ext}^1_A(Ai, S_j) = 0$ since Ai is projective. Thus we have an isomorphism of k-vector spaces

$$\mathrm{Hom}_A(J(A)i, S_j) \cong \mathrm{Ext}^1_A(S_i, S_j).$$

Now since S_j is simple and $J(A)i/J(A)^2 i$ is semisimple as A-modules, we have

$$\begin{gathered} \mathrm{Hom}_A(J(A)i, S_j) \cong \mathrm{Hom}_A(J(A)i/J(A)^2 i, S_j), \\ \dim_k \mathrm{Hom}_A(J(A)i/J(A)^2 i, S_j) = \dim_k \mathrm{Hom}_A(S_j, J(A)i/J(A)^2 i), \\ \mathrm{Hom}_A(S_j, J(A)i/J(A)^2 i) \cong \mathrm{Hom}_A(Aj, J(A)i/J(A)^2 i). \end{gathered}$$

But

$$\mathrm{Hom}_A(Aj, J(A)i/J(A)^2 i) \cong \mathrm{Hom}_A(Aj, J(A)i)/\mathrm{Hom}_A(Aj/J(A)^2 i),$$

because Aj is a projective A-module. The proposition follows from the isomorphism of k-vector spaces

$$\mathrm{Hom}_A(Ai, U) \cong iU$$

for an A-module U. □

Let $\{ \alpha_{ji}^{(r)} \mid 1 \leq r \leq n_{ji} \}$ be the set of arrows of Q from i to j. Choose elements $a_{ji}^{(r)} \in jJ(A)i$, $1 \leq r \leq n_{ji}$ such that their images in $jJ(A)i/jJ(A)^2 i$ form a k-basis for $jJ(A)i/jJ(A)^2i$. Define a k-algebra homomorphism

$$\begin{array}{rccc} \varphi\colon & kQ & \to & eAe \\ & e_i & \mapsto & i \\ & \alpha_{ji}^{(r)} & \mapsto & a_{ji}^{(r)} \end{array}$$

for every $i, j \in J$ and $1 \leq r \leq n_{ji}$. Then

THEOREM 2.20 (Gabriel). *φ is a surjective algebra homomorphism with* $\mathrm{Ker}\varphi \subseteq kQ_{(2)}$. *In other words, for every finite dimensional k-algebra A, there exists a quiver with relations (Q, I) such that A is Morita equivalent to the path algebra kQ/I of (Q, I).*

PROOF. φ is an algebra homomorphism because the only relations in kQ except linear relations are that products of non-composable paths are zero, and φ preserves those relations. By construction, φ is surjective modulo $eJ(A)^2e$; so by (2.21), φ is a surjective algebra homomorphism. Finally, since φ sends $kQ_{(2)}$ to $eJ(A)^2e$, and the e_i and $\alpha_{ji}^{(r)}$ to a k-basis for a complement of $kQ_{(2)}$ in kQ as k-vector spaces, we have that $\mathrm{Ker}\varphi \subseteq kQ_{(2)}$. □

LEMMA 2.21. *Suppose that A is a ring with a nilpotent ideal I and A' is a subring of A such that $A' + I^2 = A$. Then we have $A' = A$.*

PROOF. We show by induction that $A' + I^n = A' + I^{n+1}$ for all $n \geq 1$. Then the lemma follows from the fact that I is nilpotent. The case $n = 1$ is trivial. Suppose that $A' + I^{n-1} = A' + I^n$ for $n > 1$. We need to show that $A' + I^n = A' + I^{n+1}$, or equivalently, $I^n \subseteq A' + I^{n+1}$. Let $x \in I^{n-1}$, $y \in I$. By assumption, there exist $x' \in A' \cap I^{n-1}$ such that $x - x' \in I^n$ and $y' \in A' \cap I$ such that $y - y' \in I^2$. Then we have

$$xy = (x - x')y + x'(y - y') + x'y' \in A' + I^{n+1}.$$

□

4. Weighted Fusion Category Algebras

Let G be a finite group and let k be an algebraically closed field of prime characteristic p. Let b be a block of kG. Fix a maximal b-Brauer pair (P, e) for kG, and for each subgroup Q of P, let e_Q denote the unique block of $kC_G(Q)$ such that $(P, e) \geq (Q, e_Q)$. Let $\mathcal{F} = \mathcal{F}_{(P,e)}(G, b)$. Let d be the *defect* of the block b, i.e. the integer d such that $|P| = p^d$.

In [**26**], Linckelmann makes the following conjecture which says, roughly, that the $\alpha(Q)$ can be 'glued together' to a single second cohomology class of the category $\overline{\mathcal{F}}^c$.

CONJECTURE 2.22. *There is a unique second cohomology class $\alpha \in H^2(\overline{\mathcal{F}}^c, \underline{k}^\times)$ whose restriction to $\mathrm{Aut}_{\overline{\mathcal{F}}^c}(Q)$ is the class $\alpha(Q)$ given in (1.51) for any $\mathcal{F}$-centric subgroup Q of P.*

REMARK 2.23. In [**25**, 1.3], Linckelmann gives a criterion for the existence and uniqueness of α.

REMARK 2.24. If b is the principal block of kG, then for every $\mathcal{F}$-centric subgroup Q of P, e_Q is the principal block of $kC_G(Q)$, and hence $\overline{e}_Q$ is the principal block of $kC_G(Q)/Z(Q)$. On the other hand, $\overline{e}_Q$ has trivial defect group. Thus $H := C_G(Q)/Z(Q)$ is a p'-group, and hence $\overline{e}_Q = \frac{1}{|H|}\sum_{x\in H} x$. Therefore

$$kC_G(Q)/Z(Q)\overline{e}_Q \cong k.$$

This shows that $\alpha_Q = 0$ for every $\mathcal{F}$-centric subgroup Q of P. Thus we may take $\alpha = 0$ in this case.

DEFINITION 2.25. Let

$$\overline{\epsilon} = \sum_Q \overline{\epsilon}_Q$$

where Q runs over representatives of isomorphism classes of objects of $\overline{\mathcal{F}}^c$ and $\overline{\epsilon}_Q$ denotes the sum of blocks of $k_{\alpha(Q)}\mathrm{Aut}_{\overline{\mathcal{F}}^c}(Q)$ whose block algebras are full matrix algebras over k. Then the *weighted fusion category algebra* of the block b is

$$\overline{\mathcal{F}}(b) = \overline{\epsilon} k_\alpha \overline{\mathcal{F}}^c \overline{\epsilon}.$$

Note that $\overline{\mathcal{F}}(b)$ is uniquely defined up to Morita equivalence.

THEOREM 2.26 (Linckelmann [**26**]). *Assume that the conjecture (2.22) holds.*

(1) *Alperin's weight conjecture is equivalent to the identity*

$$\ell(b) = \ell(\overline{\mathcal{F}}(b)).$$

(2) *$\overline{\mathcal{F}}(b)$ is quasi-hereditary.*

PROOF. Let

$$A := \overline{\mathcal{F}}(b) = \overline{\epsilon} k_\alpha \overline{\mathcal{F}}^c \overline{\epsilon}$$

is the weighted fusion category algebra for the block b. Then we have a decomposition as k-vector spaces

$$A = B \oplus N$$

where

$$B = \bigoplus_{Q} \overline{\epsilon}_Q k_\alpha \mathrm{Aut}_{\overline{\mathcal{F}}^c}(Q)\overline{\epsilon}_Q,$$

$$N = \bigoplus_{Q \neq R} \overline{\epsilon}_R k_\alpha \mathrm{Hom}_{\overline{\mathcal{F}}^c}(Q, R)\overline{\epsilon}_Q,$$

where Q, R run over representatives of the isomorphism classes of objects of the category $\overline{\mathcal{F}}^c$, and $k_\alpha \mathrm{Hom}_{\overline{\mathcal{F}}^c}(Q, R)$ denotes the k-span of $\mathrm{Hom}_{\overline{\mathcal{F}}^c}(Q, R)$ in A. By definition of the $\overline{\epsilon}_Q$, B is a semisimple algebra over k. On the other hand, since $\mathrm{Hom}_{\overline{\mathcal{F}}^c}(Q, R) \neq \emptyset$ implies that $|Q| < |R|$ (if $|Q| = |R|$, then Q and R are isomorphic in the category $\overline{\mathcal{F}}^c$), we have $N^d = 0$. Thus N is the Jacobson radical of the algebra A, and the conjugacy classes of primitive idempotents of A coincide with the conjugacy classes of primitive idempotents of B. Moreover, by the definition of the $\overline{\epsilon}_Q$, we have

$$l(\overline{\epsilon}_Q k_\alpha \mathrm{Aut}_{\overline{\mathcal{F}}^c}(Q)\overline{\epsilon}_Q) = z(k_\alpha \mathrm{Aut}_{\overline{\mathcal{F}}^c}(Q)),$$

and hence

$$l(A) = \sum_{Q} z(k_\alpha \mathrm{Aut}_{\overline{\mathcal{F}}^c}(Q))$$

where Q runs over representatives of the isomorphism classes of objects of $\overline{\mathcal{F}}^c$. This, together with (1.1.51), proves (1).

For each representative Q of the isomorphism classes of objects of $\overline{\mathcal{F}}^c$, choose a set J_Q of representatives of conjugacy classes of primitive idempotents of

$$\overline{\epsilon}_Q k_\alpha \mathrm{Aut}_{\overline{\mathcal{F}}^c}(Q)\overline{\epsilon}_Q$$

so that the elements of J_Q are pairwise orthogonal. Set $J = \bigcup_Q J_Q$ where Q runs over representatives of the isomorphism classes of objects of the category $\overline{\mathcal{F}}^c$. Then J is a set of pairwise orthogonal representatives of conjugacy classes of primitive idempotents of A, which is in bijective correspondence with a set of representatives of isomorphism classes of simple A-modules, given by

$$S_j = Aj/J(A)j \quad \text{for } j \in J.$$

Now define a partial ordering $\leq$ on J so that we have $j < i$ precisely when $i \in J_Q$, $j \in J_R$, and $|Q| < |R|$. Let $i \in J_Q$, $j \in J_R$. Then

$$jJ(A)i = \begin{cases} 0, & \text{if } Q = R, \\ jk_\alpha \mathrm{Hom}_{\overline{\mathcal{F}}^c}(Q, R)i, & \text{if } Q \neq R, \end{cases}$$

and so it is zero unless $|Q| < |R|$. Thus we see that the Ext-quiver of the algebra A has vertices labeled by the elements of J, which are 'layered' by the size of their associated $\mathcal{F}$-centric subgroups, and arrows always 'going up' the layer; more precisely, for $i, j \in J$, there are arrows from i to j only when $j < i$. It follows that with respect to the partial ordering $\leq$ on J defined above, each projective indecomposable module over A is the same as the associated standard module. This proves (2). $\square$

In fact, the proof of the above theorem shows that the weighted fusion category algebra $\overline{\mathcal{F}}(b)$ belongs to a special type of quasi-hereditary algebras:

PROPOSITION 2.27. *The weighted fusion category algebra $\overline{\mathcal{F}}(b)$ for a block b is quasi-hereditary. Moreover, every standard module of $\overline{\mathcal{F}}(b)$ is projective, every costandard module is simple, and every tilting module is projective. Therefore, the Ringel dual of $\overline{\mathcal{F}}(b)$ is Morita equivalent to $\overline{\mathcal{F}}(b)$ (with the opposite order).*

PROOF. We have shown in the proof of Theorem 2.26 that the weighted fusion category algebra $\overline{\mathcal{F}}(b)$ is quasi-hereditary and every standard module of $\overline{\mathcal{F}}(b)$ is projective. It follows that every costandard module is simple by [**13**, A2.2(iv)]. Then we have that every module with Δ-filtration is projective and that every module has ∇-filtration. Thus every tilting module is projective. Thus the the Ringel dual of $\overline{\mathcal{F}}(b)$ is Morita equivalent to $\overline{\mathcal{F}}(b)$. $\square$

COROLLARY 2.28. *The weighted fusion category algebra $\overline{\mathcal{F}}(b)$ for a block b has global dimension $\leq 2(d-1)$ where d is the defect of the block b.*

PROOF. Since the trivial subgroup $\{1\}$ of P is not $\mathcal{F}$-centric, the longest possible chain in J has length $d-1$. Now the corollary follows from Proposition 2.11. $\square$

5. The Weighted Fusion Category Algebras for Tame Blocks

5.1. Tame Blocks and their Defect Groups. Let k be an algebraically closed field of prime characteristic p. The (finite dimensional) k-algebras are divided into three mutually exclusive representation types, namely, finite, tame, and wild. A k-algebra is of *finite representation type* if there are only finitely many indecomposable modules up to isomorphism; it is of *tame representation type* if, roughly speaking, it is not of finite representation type and indecomposable modules in each dimension come from finitely many one parameter families with finite exceptions; otherwise it is of *wild representation type*. For block algebras of finite groups, there are simple criteria for determining their representation types:

THEOREM 2.29 (Bondarenko and Drozd [7]). *Let G be a finite group and k an algebraically closed field of prime characteristic p. Let b be a block of kG with defect group P.*

(1) *kGb is of finite representation type if and only if P is cyclic;*
(2) *kGb is of tame representation type if and only if $p = 2$ and P is dihedral, semidihedral, or quaternion;*
(3) *kGb is of wild representation type in all other cases.*

To fix notations, let us recall the definitions of dihedral, semidihedral, and quaternion 2-groups. For every positive integer n, let C_n be the cyclic group of order n. Let x, y be generators of $\mathrm{C}_{2^{n-1}}$, C_2, respectively, for $n \geq 2$. The *dihedral group* D_{2^n} of order 2^n is the semidirect product $\mathrm{C}_{2^{n-1}} \rtimes \mathrm{C}_2$ where $yxy^{-1} = x^{-1}$. If $n \geq 4$ and $yxy^{-1} = x^{2^{n-2}-1}$, we get the *semidihedral group* SD_{2^n} of order 2^n. Finally, let $n \geq 3$, y_1 a generator of C_4, and $G = \mathrm{C}_{2^{n-1}} \rtimes \mathrm{C}_4$ where $y_1 x y_1^{-1} = x^{-1}$. One can easily check that involutions $x^{2^{n-2}}$ and y_1^2 generate the center $Z(G)$ of G. Then the quotient group $G/\langle\, x^{2^{n-2}} y_1^2 \,\rangle$ of G by the subgroup $\langle\, x^{2^{n-2}} y_1^2 \,\rangle$ generated by $x^{2^{n-2}} y_1^2$ is called the *quaternion group* Q_{2^n} of order 2^n. In summary, these three 2-groups can be presented as follows:

$$\begin{aligned}
\mathrm{D}_{2^n} &= \langle\, x,\, y \mid x^{2^{n-1}} = 1,\, y^2 = 1,\, yxy^{-1} = x^{-1} \,\rangle \quad (n \geq 2)\\
\mathrm{SD}_{2^n} &= \langle\, x,\, y \mid x^{2^{n-1}} = 1,\, y^2 = 1,\, yxy^{-1} = x^{2^{n-2}-1} \,\rangle \quad (n \geq 4)\\
\mathrm{Q}_{2^n} &= \langle\, x,\, y \mid x^{2^{n-1}} = 1,\, y^2 = x^{2^{n-2}},\, yxy^{-1} = x^{-1} \,\rangle \quad (n \geq 3)
\end{aligned}$$

(in Q_{2^n}, take as y the image of y_1). Note that these are all nonabelian groups except for the Klein four group $\mathrm{D}_4 \cong \mathrm{C}_2 \times \mathrm{C}_2$.

In a series of papers culminating in [14], Erdmann computed Morita types of all the tame block algebras. In particular, Erdmann showed that every tame block occurs as a principal block of some suitable finite group.

For later use, let us do some computations.

Let $P = \mathrm{D}_{2^n}, \mathrm{SD}_{2^n}$, or Q_{2^n}. In any case, we have

$$P - \langle\, x \,\rangle = \{\, x^j y \mid 1 \leq j < 2^{n-1} \,\},$$

and when $yxy^{-1} = x^r$,

$$(x^j y)^2 = x^j y x^j y = x^j y x^j y^{-1} y^2 = x^j x^{rj} y^2 = x^{(r+1)j} y^2$$
$$(x^j y) x^i (x^j y)^{-1} = x^j y x^i y^{-1} x^{-j} = x^j x^{ri} x^{-j} = x^{ri}$$
$$(x^i) x^j y (x^i)^{-1} = x^{i+j} y x^{-i} y^{-1} y = x^{i+j} x^{-ri} y = x^{(1-r)i+j} y$$
$$(x^i y) x^j y (x^i y)^{-1} = x^i y x^{j-i} y^{-1} y = x^i x^{r(j-i)} y = x^{(1-r)i+rj} y$$

for arbitrary exponents i, j. It follows that, for any r,

$$[x^i, x^j y] = 1 \Leftrightarrow i \equiv 0 \mod 2^{n-2}$$
$$[x^i y, x^j y] = 1 \Leftrightarrow i \equiv j \mod 2^{n-2},$$

and hence

$$C_P(x^i) = \begin{cases} P, & \text{if } i \equiv 0 \mod 2^{n-2} \\ \langle\, x \,\rangle, & \text{otherwise} \end{cases}$$
$$C_P(x^j y) = \langle\, x^{2^{n-2}},\, x^j y \,\rangle.$$

In particular, $Z(P) = \langle\, x^{2^{n-2}} \,\rangle$.

Now we analyze subgroups of P and their automorphism groups. By above computations, we see that

(1) $P = \mathrm{D}_{2^n}$: All the $x^j y$ are of order 2; proper subgroups of P are

$$\begin{aligned} \text{cyclic} : &\ \langle\, x^{2^{n-m-1}} \,\rangle \cong \mathrm{C}_{2^m} \quad (0 \le m \le n-1) \\ &\ \langle\, x^j y \,\rangle \cong C_2 \\ \text{dihedral} : &\ \langle\, x^{2^{n-m}}, x^j y \,\rangle \cong \mathrm{D}_{2^m} \quad (2 \le m \le n-1). \end{aligned}$$

(2) $P = \mathrm{SD}_{2^n}$: $x^j y$ is of order 2 if j is even, and of order 4 if j is odd; proper subgroups of P are

$$\begin{aligned} \text{cyclic} : &\ \langle\, x^{2^{n-m-1}} \,\rangle \cong \mathrm{C}_{2^m} \quad (0 \le m \le n-1) \\ &\ \langle\, x^{2j+1} y \,\rangle \cong C_4 \\ \text{dihedral} : &\ \langle\, x^{2^{n-m}}, x^{2j} y \,\rangle \cong \mathrm{D}_{2^m} \quad (2 \le m \le n-1) \\ \text{quaternion} : &\ \langle\, x^{2^{n-m}}, x^{2j+1} y \,\rangle \cong \mathrm{Q}_{2^m} \quad (3 \le m \le n-1). \end{aligned}$$

(3) $P = \mathrm{Q}_{2^n}$: All the $x^j y$ are of order 4; proper subgroups of P are

$$\begin{aligned} \text{cyclic} : \langle\, x^{2^{n-m-1}} \,\rangle &\cong \mathrm{C}_{2^m} \quad (0 \le m \le n-1) \\ \langle\, x^j y \,\rangle &\cong C_4 \\ \text{quaternion} : \langle\, x^{2^{n-m}}, x^j y \,\rangle &\cong \mathrm{Q}_{2^m} \quad (3 \le m \le n-1). \end{aligned}$$

A subgroup Q of P is called a *centric subgroup* of P if $C_P(Q) \subseteq Q$, or equivalently, $C_P(Q) = Z(Q)$. In $P = \mathrm{D}_4$, there is no centric subgroup except for D_4 itself. In all other cases, every subgroup of P is centric except for cyclic subgroups of index > 2.

PROPOSITION 2.30. *Automorphism groups of cyclic, dihedral, semidihedral, and quaternion 2-groups of order ≥ 4 are all nontrivial 2-groups except for*

$$\mathrm{Aut}(\mathrm{D}_4) \cong \Sigma_3, \qquad \mathrm{Aut}(\mathrm{Q}_8) \cong \Sigma_4$$

where Σ_n denotes the symmetric group on n letters.

PROOF. (1) C_{2^n}: We have

$$\mathrm{Aut}(\mathrm{C}_{2^n}) = \begin{cases} \mathrm{C}_{2^{n-2}} \times \mathrm{C}_2, & \text{if } n \geq 3 \\ \mathrm{C}_2, & \text{if } n = 2. \end{cases}$$

(2) D_{2^n}: $\langle\, x \,\rangle$ is the unique cyclic subgroup of order 2^{n-1} if $n \geq 3$. In this case, automorphisms of D_{2^n} must preserve $\langle\, x \,\rangle$ and send x to some x^{2i+1} and y to some $x^j y$. Conversely, every such map is an automorphism of D_{2^n} by above computation. Thus we have

$$|\mathrm{Aut}(\mathrm{D}_{2^n})| = 2^{(n-2)} 2^{(n-1)} = 2^{(2n-3)}$$

for $n \geq 3$. If $n = 2$, then all three nonidentity elements of D_4 are mutually commutative involutions, and hence $\mathrm{Aut}(\mathrm{D}_4)$ precisely consists of the permutations of those nonidentity elements. Thus $\mathrm{Aut}(\mathrm{D}_4) \cong \Sigma_3$.

(3) SD_{2^n}: Since $n \geq 4$, $\langle\, x \,\rangle$ is the unique cyclic subgroup of order 2^{n-1}. By the same argument as (2), $\mathrm{Aut}(\mathrm{SD}_{2^n})$ consists of maps of the form

$$x \mapsto x^{2i+1}, \qquad y \mapsto x^{2j+1}.$$

Thus we have $|\mathrm{Aut}(\mathrm{SD}_{2^n})| = 2^{(n-2)} 2^{(n-2)} = 2^{(2n-4)}$.

(4) Q_{2^n}: $\langle\, x \,\rangle$ is the unique cyclic subgroup of order 2^{n-1} if $n \geq 4$. In this case, by the same argument as (2), $\mathrm{Aut}(\mathrm{Q}_{2^n})$ consists of maps of the form

$$x \mapsto x^{2i+1}, \qquad y \mapsto x^j.$$

Thus we have $|\mathrm{Aut}(\mathrm{Q}_{2^n})| = 2^{(n-2)}2^{(n-1)} = 2^{(2n-3)}$ for $n \geq 4$. If $n = 3$, then one easily checks that $\mathrm{Q}_8/Z(\mathrm{Q}_8) \cong D_4$, and the canonical group homomorphism

$$\mathrm{Aut}(\mathrm{Q}_8) \to \mathrm{Aut}(\mathrm{Q}_8/Z(\mathrm{Q}_8))$$

is a surjection with kernel isomorphic to D_4. Since $\mathrm{Aut}(\mathrm{Q}_8/Z(\mathrm{Q}_8)) \cong \Sigma_3$ by (1), we conclude that $\mathrm{Aut}(\mathrm{Q}_8) \cong \Sigma_4$. $\square$

5.2. The Weighted Fusion Category Algebras for Tame Blocks. Let k be an algebraically closed field of characteristic 2. Let b be a block of kG for some finite group G with defect group P which is a dihedral, semidihedral, or quaternion 2-group. Fix a maximal b-Brauer pair (P, e), and let e_Q be the unique block of $kC_G(Q)$ such that $(P, e) \geq (Q, e_Q)$. Finally, let $\mathcal{F} = \mathcal{F}_{(P,e)}(G, b)$.

THEOREM 2.31. *Let $\overline{\mathcal{F}}(b)$ be the weighted fusion category algebra of the block b over k. Then $\overline{\mathcal{F}}(b)$ is Morita equivalent to the path algebra of one of the following quivers:*

$$\bullet^1$$

$$^2\bullet \longrightarrow \bullet^1$$

$$\bullet^1 \ \bullet^2 \ \bullet^3$$

$$^2\bullet \longrightarrow {}^1\bullet \longleftarrow \bullet^3$$

To prove Theorem 2.31, we first compute the $\overline{\mathcal{F}}$-automorphism groups of the $\mathcal{F}$-centric subgroups of P.

PROPOSITION 2.32. *Let Q be an $\mathcal{F}$-centric subgroup of P.*

(1) *If $Q < P$ and $Q \ncong \mathrm{D}_4, \mathrm{Q}_8$, then $\mathrm{Aut}_{\overline{\mathcal{F}}}(Q)$ is a nontrivial 2-group.*
(2) *If $Q < P$ and $Q \cong \mathrm{D}_4, \mathrm{Q}_8$, then $\mathrm{Aut}_{\overline{\mathcal{F}}}(Q) \cong \mathrm{C}_2$ or Σ_3.*
(3) *If $Q = P \ncong \mathrm{D}_4, \mathrm{Q}_8$, then $\mathrm{Aut}_{\overline{\mathcal{F}}}(P) = 1$.*
(4) *If $Q = P \cong \mathrm{D}_4, \mathrm{Q}_8$, then $\mathrm{Aut}_{\overline{\mathcal{F}}}(P) = 1$ or C_3.*

PROOF. $\mathcal{F}$-centric subgroups of P are of order ≥ 4 by the remark preceeding (2.30). Thus $\mathrm{Aut}(Q)$ is a nontrivial 2-group if and only if $Q \ncong \mathrm{D}_4, \mathrm{Q}_8$ by (2.30). Note that $\mathrm{Aut}_{\overline{\mathcal{F}}}(Q) \cong N_G(Q, e_Q)/QC_G(Q)$, and

$$\mathrm{Aut}_Q(Q) \leq \mathrm{Aut}_P(Q) \leq \mathrm{Aut}_{\mathcal{F}}(Q) \leq \mathrm{Aut}(Q).$$

(1), (2): $Q < P$ implies that $Q < N_P(Q)$. Thus $\mathrm{Aut}_P(Q) \cong N_P(Q)/C_P(Q)$ properly contains $\mathrm{Aut}_Q(Q) \cong Q/Z(Q)$ (note that $C_P(Q) = Z(Q)$ since Q is $\mathcal{F}$-centric). If $Q \ncong \mathrm{D}_4, \mathrm{Q}_8$, then $\mathrm{Aut}(Q)$ is a nontrivial 2-group, and it follows that $\mathrm{Aut}_{\overline{\mathcal{F}}}(Q)$ is a nontrivial 2-group. If $Q \cong \mathrm{D}_4$, then we have

$$1 = \mathrm{Aut}_Q(Q) < \mathrm{Aut}_P(Q) \leq \mathrm{Aut}_{\mathcal{F}}(Q) \leq \mathrm{Aut}(Q) \cong \Sigma_3,$$

and so we get the desired result. If $Q \cong \mathrm{Q}_8$, then we have

$$\mathrm{D}_4 \cong \mathrm{Aut}_Q(Q) < \mathrm{Aut}_P(Q) \leq \mathrm{Aut}_{\mathcal{F}}(Q) \leq \mathrm{Aut}(Q) \cong \Sigma_4.$$

Then $1 < \mathrm{Aut}_P(Q)/\mathrm{Aut}_Q(Q) \leq \mathrm{Aut}_{\overline{\mathcal{F}}}(Q) \leq \Sigma_3$, and so we get the same result.

(3), (4): $\mathrm{Aut}_{\overline{\mathcal{F}}}(P)$ is a $2'$-group by Brauer's First Main Theorem (1.33). If $P \ncong \mathrm{D}_4, \mathrm{Q}_8$, then $\mathrm{Aut}_{\overline{\mathcal{F}}}(P)$ is also a 2-group. Thus $\mathrm{Aut}_{\overline{\mathcal{F}}}(P) = 1$. If $P \cong \mathrm{D}_4$, then we have

$$1 = \mathrm{Aut}_P(P) \leq \mathrm{Aut}_{\mathcal{F}}(P) \leq \mathrm{Aut}(P) \cong \Sigma_3.$$

Since $\mathrm{Aut}_{\overline{\mathcal{F}}}(P)$ is a $2'$-group, it follows that $\mathrm{Aut}_{\overline{\mathcal{F}}}(P) = \mathrm{Aut}_{\mathcal{F}}(P) \cong 1, \mathrm{C}_3$. If $P \cong \mathrm{D}_8$, then we have

$$\mathrm{D}_4 = \mathrm{Aut}_P(P) \leq \mathrm{Aut}_{\mathcal{F}}(P) \leq \mathrm{Aut}(P) \cong \Sigma_4.$$

Then $1 \leq \mathrm{Aut}_{\overline{\mathcal{F}}}(P) \leq \Sigma_3$, and so $\mathrm{Aut}_{\overline{\mathcal{F}}}(P) \cong 1, \mathrm{C}_3$.

□

In particular, the $\overline{\mathcal{F}}$-automorphism groups of the $\mathcal{F}$-centric subgroups of P are 1, C_2, or Σ_3. Then by the following lemma, we may take $\alpha \in H^2(\overline{\mathcal{F}}^c, \underline{k}^\times)$ in Conjecture 2.22 to be zero. Moreover, $\alpha = 0$ is the unique solution to the gluing problem. We prove the uniqueness of α in the next section.

LEMMA 2.33. *If k is an algebraically closed field of characteristic 2, we have*

$$H^2(\mathrm{C}_2, k^\times) = H^2(\mathrm{C}_3, k^\times) = H^2(\Sigma_3, k^\times) = 0.$$

PROOF. We show $H^2(\Sigma_3, k^\times) = 0$. The other statement can be proven in the same way. We need to show that every extension of the group Σ_3

$$1 \to k^\times \to \widehat{\Sigma}_3 \xrightarrow{\pi} \Sigma_3 \to 1$$

by the trivial $\mathbb{Z}\Sigma_3$-module $k^\times$ splits. Identify $k^\times$ with its image in $\widehat{\Sigma}_3$. Let $\Sigma_3 = \langle\, x, y \mid x^3 = 1, y^2 = 1, yxy^{-1} = x^{-1}\,\rangle$. Choose $\widehat{x}, \widehat{y} \in \widehat{\Sigma}_3$ such that $\pi(\widehat{x}) = x$, $\pi(\widehat{y}) = y$. Then we have

$$\widehat{x}^3 = \lambda, \quad \widehat{y}^2 = \mu, \quad \widehat{y}\widehat{x}\widehat{y}^{-1} = \nu\widehat{x}^{-1}$$

for some $\lambda, \mu, \nu \in k^\times$. Raising to the third power of the third identity, we get $\lambda = \nu^3\lambda^{-1}$, or $\lambda^2 = \nu^3$. Since k is algebraically closed, there exist $\mu_1, \nu_1 \in k^\times$ such that $\mu_1^2 = \mu^{-1}$, $\nu_1^2 = \nu^{-1}$. Note that $\nu_1^6 = \nu^{-3} = \lambda^{-2}$, and so $\nu_1^3 = \lambda^{-1}$ because char $k = 2$. Then it follows that $\pi(\nu_1 \widehat{x}) = x$, $\pi(\mu_1 \widehat{y}) = y$, and

$$(\nu_1 \widehat{x})^3 = 1, \quad (\mu_1 \widehat{y})^2 = 1, \quad (\mu_1 \widehat{y})(\nu_1 \widehat{x})(\mu_1 \widehat{y})^{-1} = (\nu_1 \widehat{x})^{-1},$$

showing that the extension splits. $\square$

Now we locate proper $\mathcal{F}$-centric subgroups Q of P with $\overline{\epsilon}_Q \neq 0$ up to P-conjugacy. Note that if $\mathrm{Aut}_{\overline{\mathcal{F}}}(Q)$ is a nontrivial p-group, then $k\mathrm{Aut}_{\overline{\mathcal{F}}}(Q)$ is a local algebra, so its unique block $1_{\mathrm{Aut}_{\overline{\mathcal{F}}}(Q)}$ is the principal block with defect group $\mathrm{Aut}_{\overline{\mathcal{F}}}(Q)$, hence $\overline{\epsilon}_Q = 0$. By inspection of the subgroup list in §5.1 and Proposition 2.32, we get:

(1) $P = \mathrm{D}_{2^n}$ has subgroups isomorphic to D_4, but no subgroups isomorphic to Q_8; subgroups isomorphic to D_4 are P-conjugate to

$$\langle\, x^{2^{n-2}},\, y \,\rangle \quad \textit{or} \quad \langle\, x^{2^{n-2}},\, xy \,\rangle.$$

(2) $P = \mathrm{SD}_{2^n}$ has subgroups isomorphic to D_4 and Q_4; subgroups isomorphic to D_4 are P-conjugate to

$$\langle\, x^{2^{n-2}},\, y \,\rangle;$$

subgroups isomorphic to Q_8 are P-conjugate to

$$\langle\, x^{2^{n-3}},\, xy \,\rangle.$$

(3) $P = \mathrm{Q}_{2^n}$ has subgroups isomorphic to Q_8, but no subgroups isomorphic to D_4; subgroups isomorphic to Q_8 are P-conjugate to

$$\langle\, x^{2^{n-3}},\, y \,\rangle \quad \textit{or} \quad \langle\, x^{2^{n-3}},\, xy \,\rangle.$$

Moreover, all the subgroups listed above are $\mathcal{F}$-centric subgroups. In each case, let us call those two listed subgroups basic subgroups of P. Note that a basic subgroup Q of P is $\mathcal{F}$-essential if and only if $\mathrm{Aut}_{\overline{\mathcal{F}}}(Q) \cong \Sigma_3$.

Using this information, we can compute the Morita types of the weighted fusion category algebras of tame blocks.

Case I: $P \cong \mathrm{D}_4, \mathrm{Q}_8$.

In this case, P itself is the only $\mathcal{F}$-centric subgroup of P, and $\mathrm{Aut}_{\overline{\mathcal{F}}}(P) \cong 1$ or C_3. Thus we have

$$\overline{\mathcal{F}}(b) = \overline{\epsilon}_P k\mathrm{Aut}_{\overline{\mathcal{F}}}(P)\overline{\epsilon}_P \cong k \text{ or } k \times k \times k.$$

Case II: $P \ncong \mathrm{D}_4, \mathrm{Q}_8$

Subcase 1: Both basic subgroups are not $\mathcal{F}$-essential.

In this case, we have $\overline{\epsilon}_Q = 0$ unless $Q = P$, and $\mathrm{Aut}_{\overline{\mathcal{F}}}(P) \cong 1$. Thus we have

$$\overline{\mathcal{F}}(b) = \overline{\epsilon}_P k\mathrm{Aut}_{\overline{\mathcal{F}}}(P)\overline{\epsilon}_P \cong k.$$

Subcase 2: Only one of the basic subgroups is $\mathcal{F}$-essential.

Let Q denote the $\overline{\mathcal{F}}$-essential basic subgroup. Then we have

$$k\mathrm{Aut}_{\overline{\mathcal{F}}}(Q) \cong k\Sigma_3 \cong k\mathrm{C}_2 \times M_2(k)$$

where the blocks corresponding to $k\mathrm{C}_2$, $M_2(k)$ are $(1)+(123)+(132)$, $(123)+(132)$, respectively. Thus

$$\overline{\epsilon}_Q = (123) + (132).$$

Also, the element of $k\mathrm{Aut}_{\overline{\mathcal{F}}}(Q)$ corresponding to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(k)$ is

$$j = (1) + (132) + (12) + (13).$$

Since all subgroups of P isomorphic to Q are P-conjugates of Q, we may consider only Q among all its isomorphs. Then we have $\overline{\epsilon} = \overline{\epsilon}_P + \overline{\epsilon}_Q$ and so

$$\overline{\mathcal{F}}(b) = \overline{\epsilon}_P k\overline{\mathcal{F}}^c\overline{\epsilon}_P \oplus \overline{\epsilon}_Q k\overline{\mathcal{F}}^c\overline{\epsilon}_Q \oplus \overline{\epsilon}_P k\overline{\mathcal{F}}^c\overline{\epsilon}_Q$$

Since

$$\begin{aligned} \overline{\epsilon}_P k\overline{\mathcal{F}}^c\overline{\epsilon}_P &\cong k \\ \overline{\epsilon}_Q k\overline{\mathcal{F}}^c\overline{\epsilon}_Q &\cong M_2(k) \\ (\overline{\epsilon}_P k\overline{\mathcal{F}}^c\overline{\epsilon}_Q)^2 &= 0, \end{aligned}$$

we have $\overline{\epsilon}_P k\overline{\mathcal{F}}^c\overline{\epsilon}_Q = J(\overline{\mathcal{F}}(b))$ and there are exactly two nonisomorphic simple $\overline{\mathcal{F}}(b)$-modules

$$S_1 = \overline{\mathcal{F}}(b)\overline{\epsilon}_P/J(\overline{\mathcal{F}}(b))\overline{\epsilon}_P, \qquad S_2 = \overline{\mathcal{F}}(b)j/J(\overline{\mathcal{F}}(b))j.$$

Now we need to compute $\overline{\epsilon}_P k\overline{\mathcal{F}}^c\overline{\epsilon}_Q = \overline{\epsilon}_P k\mathrm{Hom}_{\overline{\mathcal{F}}}(Q,P)\overline{\epsilon}_Q$. Note that

$$\mathrm{Hom}_{\overline{\mathcal{F}}}(Q,P) = \mathrm{Aut}_P(Q)\backslash\mathrm{Aut}_{\overline{\mathcal{F}}}(Q).$$

Since $\mathrm{Aut}_P(Q) \cong \mathrm{C}_2$, take (12) as its generator. Then

$$\mathrm{Hom}_{\overline{\mathcal{F}}}(Q,P) = \{\, \overline{(1)}, \overline{(123)}, \overline{(132)} \,\}$$

where the equivalence relation is given by $\overline{(1)} = \overline{(12)}$. Then $\overline{\epsilon}_P k\overline{\mathcal{F}}^c\overline{\epsilon}_Q$ has as a k-basis

$$\{\, \overline{(1)} + \overline{(123)}, \overline{(1)} + \overline{(132)} \,\}.$$

Then

$$J(\overline{\mathcal{F}}(b))j/J(\overline{\mathcal{F}}(b))^2 j = k(\overline{(1)} + \overline{(132)}) \cong S_1$$

Thus we have the following quiver for $\overline{\mathcal{F}}(b)$

$$ {}^{2}\bullet \longrightarrow \bullet^{1} $$

Subcase 3: The two basic subgroups are $\mathcal{F}$-isomorphic and $\mathcal{F}$-essential.

We get the same result as Subcase 3.

Subcase 4: The two basic subgroups are not $\mathcal{F}$-isomorphic and both are $\mathcal{F}$-essential.

Then we get another copy of the previous quiver, so

$$ {}^{2}\bullet \longrightarrow {}^{1}\bullet \longleftarrow \bullet^{3} $$

5.3. The existence and uniqueness of α for the tame block case. Let us keep the notations of the previous section. We show

THEOREM 2.34. $H^2(\overline{\mathcal{F}}^c, \underline{k}^\times) = 0$.

By [**25**, 11.2], it suffices to show that $H^2(\mathcal{F}^c, \underline{k}^\times) = 0$.

Let $S(\mathcal{F}^c)$ be the category defined as follows: The objects of $S(\mathcal{F}^c)$ are chains

$$\sigma = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} X_m$$

of objects X_i ($0 \leq i \leq m$), and of nonisomorphisms φ_i ($0 \leq i \leq m-1$, and where m is a nonnegative integer. A morphism in $S(\mathcal{F}^c)$ from such a chain of nonisomorphisms

$$\sigma = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} X_m$$

to a chain of nonisomorphisms

$$\tau = Y_0 \xrightarrow{\psi_0} Y_1 \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_{n-1}} Y_m$$

is a family $\mu = (\mu_i)_{0\leq i\leq m}$ where for each i there is $\omega(i) \in \{0, 1, \ldots, n\}$ such that $\mu_i \colon X_i \to Y_{\omega(i)}$ is an isomorphism such that

$$\mu_{i+1} \circ \varphi_i = \psi_{\omega(i+1)-1} \circ \cdots \circ \psi_{\omega(i)+1} \circ \psi_{\omega(i)} \circ \mu_i$$

for any $i \in \{0, 1, \ldots, m-1\}$. Let $[S(\mathcal{F}^c)]$ be the poset of isomorphism classes of $S(\mathcal{F}^c)$; more precisely, for each object σ of $S(\mathcal{F}^c)$ let $[\sigma]$ denote its isomorphism classs in $S(\mathcal{F}^c)$, and say $[\sigma] \leq [\tau]$ if $\mathrm{Hom}_{S(\mathcal{F}^c)}(\sigma, \tau) \neq \emptyset$. Then, by [**25**, 10.1, 10.5], it suffices to show that

$$H^1([S(\mathcal{F}^c)], \mathcal{A}^1) = 0$$

where $\mathcal{A}^1 \colon [S(\mathcal{F}^c)] \to \mathrm{Mod}(\mathbb{Z})$ is a covariant functor sending $[\sigma]$ to

$$H^1(\mathrm{Aut}_{S(\mathcal{F}^c)}(\sigma), k^\times) \cong \mathrm{Hom}(\mathrm{Aut}_{S(\mathcal{F}^c)}(\sigma), k^\times).$$

Since char $k = 2$, for any group G we have $\mathrm{Hom}(G, k^\times) \cong \mathrm{Hom}(G/O^{2'}(G), k^\times)$. In particular, $\mathrm{Hom}(C_2, k^\times) = \mathrm{Hom}(\Sigma_3, k^\times) = 0$. Thus, from Proposition 2.32, we get

COROLLARY 2.35. *Let Q be an $\mathcal{F}$-centric subgroup of P.*

$$\mathrm{Hom}(\mathrm{Aut}_{\mathcal{F}}(Q), k^\times) \cong \begin{cases} \mathbb{Z}/3, & \textit{if } Q = P \cong C_2 \times C_2, \mathcal{F} = \mathcal{F}_P(P \rtimes C_3) \\ \mathbb{Z}/3, & \textit{if } Q = P \cong Q_8, \mathcal{F} = \mathcal{F}_P(P \rtimes C_3) \\ 0, & \textit{otherwise.} \end{cases}$$

Let $\mathcal{C} = [S(\mathcal{F}^c)]$.

<u>Case 1</u>: $P \not\cong C_2 \times C_2, Q_8$, or $\mathcal{F} = \mathcal{F}_P(P)$. Then $\mathcal{A}^1 = 0$. Thus $H^1(\mathcal{C}, \mathcal{A}^1) = 0$.

<u>Case 2</u>: $P \cong C_2 \times C_2$, $\mathcal{F} = \mathcal{F}_P(P \rtimes C_3)$. Then $\mathcal{F}^c$, and hence $\mathcal{C}$ has one object. Thus $H^1(\mathcal{C}, \mathcal{A}^1) = 0$.

<u>Case 3</u>: $P \cong Q_8$, $\mathcal{F} = \mathcal{F}_P(P \rtimes C_3)$. Then $\mathcal{C}$ has three objects with two nonisomorphisms:

$$a{=}[Q] \bullet \xrightarrow{\ \alpha\ } \bullet\, c{=}[Q{\to}P] \xleftarrow{\ \beta\ } \bullet\, b{=}[P]$$

where $Q \cong C_4$, and

$$\mathrm{Aut}_{S(\mathcal{F}^c)}(Q) = 0, \quad \mathrm{Aut}_{S(\mathcal{F}^c)}(P) \cong \mathbb{Z}/3, \quad \mathrm{Aut}_{S(\mathcal{F}^c)}(Q \to P) = 0.$$

Thus, viewed as a $\mathbb{Z}\mathcal{C}$-module, $\mathcal{A}^1 \cong (\mathbb{Z}/3)x$ with

$$a \cdot x = c \cdot x = \alpha \cdot x = \beta \cdot x = 0, \quad b \cdot x = x.$$

Let $\mathcal{P}$ be the standard resolution of the constant functor $\underline{\mathbb{Z}}$; explicitly

$$\mathcal{P}_0 = \mathbb{Z}\mathcal{C} = \mathbb{Z}\mathcal{C}(a) \oplus \mathbb{Z}\mathcal{C}(b) \oplus \mathbb{Z}\mathcal{C}(c),$$
$$\mathcal{P}_1 = \mathbb{Z}\mathcal{C}(aa) \oplus \mathbb{Z}\mathcal{C}(bb) \oplus \mathbb{Z}\mathcal{C}(cc) \oplus \mathbb{Z}\mathcal{C}(c\alpha) \oplus \mathbb{Z}\mathcal{C}(c\beta),$$
$$\mathcal{P}_2 = \mathbb{Z}\mathcal{C}(aaa) \oplus \mathbb{Z}\mathcal{C}(bbb) \oplus \mathbb{Z}\mathcal{C}(ccc) \oplus \mathbb{Z}\mathcal{C}(cc\alpha) \oplus \mathbb{Z}\mathcal{C}(cc\beta) \oplus \mathbb{Z}\mathcal{C}(c\alpha a) \oplus \mathbb{Z}\mathcal{C}(c\beta b),$$

where a, b, c denote the identity maps on themselves, respectively. Then

$$\mathrm{Hom}_{\mathbb{Z}\mathcal{C}}(\mathcal{P}_0, \mathcal{A}^1) \cong (\mathbb{Z}/3)\underline{b},$$
$$\mathrm{Hom}_{\mathbb{Z}\mathcal{C}}(\mathcal{P}_1, \mathcal{A}^1) \cong (\mathbb{Z}/3)\underline{bb},$$
$$\mathrm{Hom}_{\mathbb{Z}\mathcal{C}}(\mathcal{P}_2, \mathcal{A}^1) \cong (\mathbb{Z}/3)\underline{bbb},$$

where $\underline{b}(b) = x$, $\underline{bb}(bb) = x$, $\underline{bbb}(bbb) = x$ (and they are zero on other elements). Let δ be the coboundary map. Then

$$\delta\underline{b}(aa) = \delta\underline{b}(bb) = \delta\underline{b}(cc) = 0,$$
$$\delta\underline{b}(c\alpha) = \alpha\underline{b}(a) - \underline{b}(c) = 0,$$
$$\delta\underline{b}(c\beta) = \beta\underline{b}(b) - \underline{b}(c) = \beta x = 0.$$

Thus $\delta\underline{b} = 0$. Also

$$\delta\underline{bb}(aaa) = \delta\underline{bb}(ccc) = \delta\underline{bb}(cc\alpha) = \delta\underline{bb}(cc\beta) = \delta\underline{bb}(c\alpha a) = 0,$$
$$\delta\underline{bb}(bbb) = \underline{bb}(bb) - \underline{bb}(bb) + \underline{bb}(bb) = x,$$
$$\underline{bb}(c\beta b) = \underline{bb}(\beta b) - \underline{bb}(c\beta) + \underline{bb}(c\beta) = \beta\underline{bb}(bb) = \beta x = 0.$$

Thus $\delta\underline{bb} = \underline{bbb}$. Therefore,

$$Z^1(\mathcal{C}, \mathcal{A}^1) = B^1(\mathcal{C}, \mathcal{A}^1) = 0,$$

and hence $H^1(\mathcal{C}, \mathcal{A}^1) = 0$.

CHAPTER 3

The Weighted Fusion Category Algebra for the general linear group and the q-Schur Algebra

We consider the weighted fusion category algebra for principal blocks of $\mathrm{GL}_n(q)$. They are quasi-hereditary algebras canonically associated with $\mathrm{GL}_n(q)$ giving representation theoretic information of $\mathrm{GL}_n(q)$.

In fact, there is another such algebra associated with $\mathrm{GL}_n(q)$, the q-Schur algebra. The q-Schur algebra was introduced to compute another important representation theoretic invariant of $\mathrm{GL}_n(q)$, the decomposition matrix, which describes the relation between the ordinary representations and the modular representations of $\mathrm{GL}_n(q)$. So one may conjecture that there are certain relations between them.

We compute the Morita types of some low rank weighted fusion category algebras for principal 2-blocks of $\mathrm{GL}_n(q)$ for q odd and compare them with those of the q-Schur algebra given by Erdmann and Nakano **[15]**. It turns out that the weighted fusion category algebra $\overline{\mathcal{F}}(b_0)$ of the principal 2-block b_0 of $\mathrm{GL}_2(q)$ is Morita equivalent to the quotient of $\mathcal{S}_2(q)$ by its socle. Especially, this gives a canonical bijection between weights for the principal 2-block b_0 of $\mathrm{GL}_2(q)$ and the simple $k\mathrm{GL}_2(q)$-modules where char $k = 2$. When $n = 3$, we don't have such a direct relation; but one can show that there is a certain pullback diagram involving those two. These results are interesting because the definition of the q-Schur algebra $\mathcal{S}_n(q)$ does not involve the p-local structure of $\mathrm{GL}_n(q)$.

1. The Weighted Fusion Category Algebra for $\mathrm{GL}_n(q)$, $n = 2, 3$

PROPOSITION 3.1. *Let k be an algebraically closed field of characteristic 2 and q an odd prime power.*

(1) *The weighted fusion category algebra $\overline{\mathcal{F}}(b_0)$ over k of the principal 2-block b_0 of $\mathrm{GL}_2(q)$ is Morita equivalent to the path algebra of the quiver*

$$\overset{1}{\bullet} \longleftarrow \overset{2}{\bullet}$$

(2) *The weighted fusion category algebra $\overline{\mathcal{F}}(b_0)$ over k of the principal 2-block b_0 of $\mathrm{GL}_3(q)$ is Morita equivalent to the path algebra of the quiver*

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longleftarrow \overset{3}{\bullet}$$

1.1. Proof of Proposition 3.1 when $n = 2$, $q \equiv 3 \mod 4$. Let $G = \mathrm{GL}_2(q)$ and q a prime power such that $q \equiv 3 \mod 4$. Let 2^{m-2} be the highest 2-power dividing $q + 1$, and let ξ be a primitive 2^{m-1}th root of unity in $\mathbb{F}_{q^2}$. Note that $m \geq 4$. Let $a = \xi + \xi^q$. Then the subgroup P of G generated by

$$x = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}, \qquad t = \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix}$$

is a Sylow 2-subgroup of G. One immediately checks that x and t are of order 2^{m-1} and 2 respectively, and

$$txt = x^{2^{m-2}-1}.$$

In other words, P is the semidiheral group SD_{2^m} of order 2^m.

Let $\mathcal{F} = \mathcal{F}_P(G)$. Then the $\mathcal{F}$-centric subgroups of P are as follows:

(1) $\mathrm{C}_2 \times \mathrm{C}_2 \cong \langle x^{2^{m-2}}, tx^{2i} \rangle$
(2) $\mathrm{D}_{2^k} \cong \langle x^{2^{m-k}}, tx^{2i} \rangle$ where $3 \leq k \leq m - 1$
(3) $\mathrm{Q}_{2^k} \cong \langle x^{2^{m-k}}, tx^{2i+1} \rangle$ where $3 \leq k \leq m - 1$
(4) $\mathrm{C}_{2^{m-1}} \cong \langle x \rangle$
(5) P

Recall that the automorphism groups of cyclic, dihedral, semidihedral, and (generalized) quaternion 2-groups of order ≥ 4 are all nontrivial 2-groups except for

$$\mathrm{Aut}(\mathrm{C}_2 \times \mathrm{C}_2) \cong \Sigma_3, \qquad \mathrm{Aut}(\mathrm{Q}_8) \cong \Sigma_4.$$

So the $\overline{\mathcal{F}}$-automorphism group of an $\mathcal{F}$-centric subgroup R of P of type (2), (3) with $k > 3$, (4), or (5) is a (possibly trivial) 2-group. If $R = P$, then since $\mathrm{Aut}_{\overline{\mathcal{F}}}(P)$ is also a $2'$-group, we have $\mathrm{Aut}_{\overline{\mathcal{F}}}(P) = \{1\}$. Thus $\bar{\epsilon}_P = 1$. If $R < P$, then we have $\mathrm{Inn}(R) < \mathrm{Aut}_{\mathcal{F}}(R)$, and hence $\mathrm{Aut}_{\overline{\mathcal{F}}}(R)$ is a nontrivial 2-group. Therefore $\bar{\epsilon}_R = 0$.

Also, since $x^{2^{m-2}} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in Z(G)$ and tx^{2i}, $-tx^{2i}$ are G-conjugate, the $\overline{\mathcal{F}}$-automorphism group of a Klein four subgroup of P is isomorphic to C_2. Thus it remains to consider the quaternion subgroups of order 8. Set

$$Q_i = \langle x^{2^{m-3}}, tx^{2i+1} \rangle, \quad i = 0, 1, \ldots, 2^{m-4} - 1.$$

First observe that all Q_i are P-conjugate. Indeed, for each pair of indices i, j, let $k = (2^{m-3} - 1)(j - i)$. Then

$$x^k t x^{2i+1} x^{-k} = t x^{(2^{m-2}-1)k} x^{2i+1-k} = t x^{2j+1}.$$

So it suffices to consider only $Q := Q_0 = \langle x^{2^{m-3}}, tx \rangle$. We have $\mathrm{Aut}_P(Q) \cong \mathrm{D}_8$ and $\mathrm{Aut}(Q) \cong \Sigma_4$. Thus $\mathrm{Aut}_{\mathcal{F}}(Q)$ is either $\mathrm{Aut}_P(Q)$ or $\mathrm{Aut}(Q)$. Since $x^{2^{m-3}}$ and tx are G-conjugate and the $\mathcal{F}$-automorphism of Q induced by that G-conjugation does not belong to $\mathrm{Aut}_P(Q)$, we conclude that

$$\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}(Q) \cong \Sigma_4.$$

Now $\mathrm{Aut}_{\overline{\mathcal{F}}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$ and $\mathrm{Aut}_Q(Q) \cong \mathrm{C}_2 \times \mathrm{C}_2$. Thus

$$\mathrm{Aut}_{\overline{\mathcal{F}}}(Q) \cong \Sigma_3.$$

Under an algebra isomorphism $k\Sigma_3 \cong k\mathrm{C}_2 \times M_2(k)$, one finds that

$$j = (1) + (132) + (12) + (13) \in k\Sigma_3$$

corresponds to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(k)$, so j is a primitive idempotent associated to Q.

Set $A := \bar{\epsilon} k \overline{\mathcal{F}}^c \bar{\epsilon}$ where $\bar{\epsilon} = \bar{\epsilon}_P + \bar{\epsilon}_Q$. Then A is Morita equivalent to $\overline{\mathcal{F}}(b_0)$. Now $J(A)^2 = 0$ and

$$\mathrm{Hom}_{\overline{\mathcal{F}}}(Q, P) \cong \mathrm{Aut}_{\overline{P}}(Q) \backslash \mathrm{Aut}_{\overline{\mathcal{F}}}(Q)$$

where $\mathrm{Aut}_{\overline{P}}(Q) := \mathrm{Aut}_P(Q)/\mathrm{Inn}(Q)$. Since $\mathrm{Aut}_{\overline{P}}(Q) \cong \mathrm{C}_2$, we may take (12) as its generator. Then

$$\mathrm{Hom}_{\overline{\mathcal{F}}}(Q, P) = \{\, \overline{(1)}, \overline{(123)}, \overline{(132)} \,\}$$

where $\overline{\sigma}$ denotes the $\mathrm{Aut}_{\overline{P}}(Q)$-orbit of $\sigma \in \mathrm{Aut}_{\overline{\mathcal{F}}}(Q)$. Then

$$J(A)j = k\mathrm{Hom}_{\overline{\mathcal{F}}}(Q, P)j = k\{\, \overline{(1)} + \overline{(132)} \,\}.$$

Thus A has the quiver with two vertices labeled by Q and P, and one arrow from Q to P.

1.2. Proof of Proposition 3.1 when $n = 2$, $q \equiv 1 \mod 4$. Let 2^m be the highest 2-power dividing $q - 1$, and let η be a primitive 2^mth root of unity in $\mathbb{F}_q$. Note that $m \geq 2$. Then the subgroup P of G generated by

$$x = \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a Sylow 2-subgroup of G. Since x, y commute and $txt = y$, we see that $P \cong \mathrm{C}_{2^m} \wr \mathrm{C}_2$. Note that $Z_0 := Z(P) = Z(G) \cap P = \langle xy \rangle \cong \mathrm{C}_{2^m}$.

Let $\mathcal{F} = \mathcal{F}_P(G)$. Then the $\mathcal{F}$-centric subgroups of P are as follows:

(1) $\langle x, y\rangle$
(2) $\langle xy, tx^i\rangle$ where $\eta^i \neq \eta^{2j}$ for any integer j
(3) $\langle xy, x^{2^i}, tx^j\rangle$ where $0 \leq i \leq m-1, 0 \leq j < 2^i$

Let R be an $\mathcal{F}$-centric subgroup of P. If $R = \langle x, y\rangle$, then we have

$$\mathrm{Aut}_{\overline{\mathcal{F}}}(R) \cong N_G(R)/RC_G(R) = R\Sigma_2/R \cong \Sigma_2,$$

where Σ_2 is viewed as the subgroup of the permutation matrices in G.

Now suppose that R is of type (2) or (3). Since $Z_0 \subseteq Z(G)$, elements of Z_0 are fixed by any $\mathcal{F}$-morphism. So every $\mathcal{F}$-automorphism of R induces an automorphism of R/Z_0, giving rise to a surjective group homomorphism

$$\Phi : \mathrm{Aut}_{\mathcal{F}}(R) \twoheadrightarrow \mathrm{Aut}_{G/Z_0}(R/Z_0).$$

Note that the kernel $\mathrm{Ker}(\Phi)$ of Φ is isomorphic to a subgroup of $\mathrm{Hom}(R, Z_0)$ whose multiplication is given by pointwise multiplication. In particular $\mathrm{Ker}(\Phi)$ is an abelian 2-group.

If R is of type (2), then $R/Z_0 \cong \mathrm{C}_2$, so $\mathrm{Aut}(R/Z_0) = \{1\}$. One can easily check that $\mathrm{Ker}(\Phi) \cong \mathrm{C}_2$ in this case. Since R is abelian, it follows that $\mathrm{Aut}_{\overline{\mathcal{F}}}(R) \cong \mathrm{C}_2$.

Suppose that R is of type (3). Then R/Z_0 is a dihedral 2-group of order ≥ 4; it is of order 4 (i.e. a Klein four group) if and only if $i = m-1$. So if $i \neq m-1$, then R/Z_0 is a dihedral 2-group of order ≥ 8, and hence its automorphism group is a (nontrivial) 2-group. Thus $\mathrm{Aut}_{\mathcal{F}}(R)$ is a 2-group. Now if $R < P$, then $\mathrm{Inn}(R) < \mathrm{Aut}_{\mathcal{F}}(R)$, so $\mathrm{Aut}_{\overline{\mathcal{F}}}(R)$ is a nontrivial 2-group; if $R = P$, then $\mathrm{Aut}_{\overline{\mathcal{F}}}(P)$ is also a $2'$-group, and hence $\mathrm{Aut}_{\overline{\mathcal{F}}}(P) = 1$.

Finally, let R be of type (3) with $i = m-1$. There are two P-conjugacy classes among these $\mathcal{F}$-centric subgroups. Indeed, for any j,

$$\langle xy, x^{2^{m-1}}, tx^j\rangle \cong \langle xy, x^{2^{m-1}}, tx^{j+2}\rangle$$

because $x^{-1}(tx^{j+1}y)x = tx^{j+2}$. Set

$$R_1 = \langle xy, x^{2^{m-1}}, t\rangle, \qquad R_2 = \langle xy, x^{2^{m-1}}, tx\rangle.$$

Since $R_i/Z_0 (i = 1, 2)$ is a Klein four group, its full automorphism group is isomorphic to Σ_3, permuting its three nonidentity elements. Those three nonidentity elements of R_1/Z_0 are all G-conjugate; in R_2/Z_0, txZ_0 and $tx^{2^{m-1}+1}Z_0$ are G-conjugate but $x^{2^{m-1}}Z_0$ is not G-conjugate to these two. For both $i = 1, 2$, we have

$\mathrm{Ker}(\Phi) = \mathrm{Inn}(R_i) \cong \mathrm{C}_2 \times \mathrm{C}_2$. Thus

$$\mathrm{Aut}_{\overline{\mathcal{F}}}(R_1) \cong \Sigma_3, \qquad \mathrm{Aut}_{\overline{\mathcal{F}}}(R_2) \cong \mathrm{C}_2.$$

Therefore we get the same quiver as in Section 1.1.

1.3. Proof of Proposition 3.1 when $n = 3$**,** $q \equiv 3 \mod 4$**.** Let $G = \mathrm{GL}_3(q)$ where q is a prime power such that $q \equiv 3 \mod 4$. Let 2^{m-2} be the highest 2-power dividing $q+1$, and let ξ be a primitive 2^{m-1}th root of unity in $\mathbb{F}_{q^2}$. Note that $m \geq 4$. Let $a = \xi + \xi^q$. Then the subgroup P of G generated by

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad t = \begin{pmatrix} 1 & a & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is a Sylow 2-subgroup of G. We have $P = \langle x, t \rangle \times \langle u \rangle \cong \mathrm{SD}_{2^m} \times \mathrm{C}_2$. Note that $Z(P) = \langle x^{2^{m-2}}, u \rangle$ and $Z_1 := Z(G) \cap P = \langle x^{2^{m-2}} u \rangle \cong \mathrm{C}_2$.

Let G_0 be $\mathrm{GL}_2(q)$ embedded in $\mathrm{GL}_3(q)$ in the upper left 2×2 minor. Then $P_0 = \langle x, t \rangle$ is a Sylow p-subgroup of G_0. Let $\mathcal{F} = \mathcal{F}_P(G)$ and $\mathcal{F}_0 = \mathcal{F}_{P_0}(G_0)$. The $\mathcal{F}$-centric subgroups of P are of the form $R_0 \times \langle u \rangle$ where R_0 is an $\mathcal{F}_0$-centric subgroups of G_0. Thus they are as follows:

(1) $\mathrm{C}_2 \times \mathrm{C}_2 \times \mathrm{C}_2 \cong \langle x^{2^{m-2}}, tx^{2i}, u \rangle$
(2) $\mathrm{D}_{2^k} \times \mathrm{C}_2 \cong \langle x^{2^{m-k}}, tx^{2i}, u \rangle$ where $3 \leq k \leq m-1$
(3) $\mathrm{Q}_{2^k} \times \mathrm{C}_2 \cong \langle x^{2^{m-k}}, tx^{2i+1}, u \rangle$ where $3 \leq k \leq m-1$
(4) $\mathrm{C}_{2^{m-1}} \times \mathrm{C}_2 \cong \langle x, u \rangle$
(5) P

Let R be an $\mathcal{F}$-centric subgroup of P. Since $Z_1 \subseteq Z(G)$, elements of Z_1 are fixed by any $\mathcal{F}$-morphism. So every $\mathcal{F}$-automorphism of R induces an automorphism of R/Z_1, giving rise to a surjective group homomorphism

$$\Phi\colon \mathrm{Aut}_{\mathcal{F}}(R) \twoheadrightarrow \mathrm{Aut}_{G/Z_1}(R/Z_1).$$

Let us show that Φ is in fact an isomorphism. $\mathrm{Ker}(\Phi)$ consists of $\mathcal{F}$-automorphisms of R sending $r \in R$ to $\pm r$. Suppose that $\alpha \in \mathrm{Ker}(\Phi)$ and $\alpha(r) = -r$ for some $r \in R$. Since r and $-r$ are G-conjugate and either $X - 1$ or $X + 1$ is an elementary divisor of r, it follows that both $X - 1$ and $X + 1$ are elementary divisors of r. Then the remaining elementary divisor is of the form $X - a$ for some $a \in \mathbb{F}_q - \{0\}$, and hence also $X + a$ is an elementary divisor of r. So we must have $a = -a$, a contradiction. Thus $\mathrm{Ker}(\Phi) = \{\mathrm{id}_R\}$ and hence Φ is an isomorphism of groups.

If R is of type (2), (3) with $k > 3$, (4), or (5), then $\mathrm{Aut}(R/Z_1)$ is a 2-group and so is $\mathrm{Aut}_{\mathcal{F}}(R)$. If $R = P$, then $\mathrm{Aut}_{\overline{\mathcal{F}}}(P)$ is a $2'$-group, so $\mathrm{Aut}_{\overline{\mathcal{F}}}(P) = \{1\}$, so $\bar{\epsilon}_P = 1$. If $R < P$, we have $\mathrm{Inn}(R) < \mathrm{Aut}_{\mathcal{F}}(R)$. Hence $\mathrm{Aut}_{\overline{\mathcal{F}}}(R)$ is a nontrivial 2-group. So $\bar{\epsilon}_R = 0$.

Let $Q := \langle x^{2^{m-3}}, tx, u\rangle \cong \mathrm{D}_8 \times \mathrm{C}_2$. All other $\mathcal{F}$-centrics of type (3) with $k = 3$ are P-conjugate to Q. Since $Q/Z_1 \cong \langle x^{2^{m-3}}, tx\rangle \leq G_0$, by the same argument as in Section 1.1 we get

$$\mathrm{Aut}_{\overline{\mathcal{F}}}(Q) \cong \Sigma_3.$$

Now let $V := \langle x^{2^{m-2}}, t, u\rangle$. Again, all the other $\mathcal{F}$-centrics of type (1) are P-conjugate to V. Then $V/Z_1 = \langle x^{2^{m-2}}Z_1, tZ_1\rangle \cong \mathrm{C}_2 \times \mathrm{C}_2$ and all three nonidentity elements of V/Z_1 are G-conjugates. Thus

$$\mathrm{Aut}_{\overline{\mathcal{F}}}(V) = \mathrm{Aut}_{\mathcal{F}}(V) = \mathrm{Aut}_{G/Z_1}(R/Z_1) \cong \Sigma_3.$$

Since Q does not contain any $\mathcal{F}$-conjugate of V, we have $\mathrm{Hom}_{\overline{\mathcal{F}}}(V, Q) = \emptyset$. On the other hand,

$$\mathrm{Hom}_{\overline{\mathcal{F}}}(Q, P) \cong \mathrm{Aut}_{\overline{P}}(Q)\backslash\mathrm{Aut}_{\overline{\mathcal{F}}}(Q) \cong C_2\backslash\Sigma_3,$$
$$\mathrm{Hom}_{\overline{\mathcal{F}}}(V, P) \cong \mathrm{Aut}_{\overline{P}}(V)\backslash\mathrm{Aut}_{\overline{\mathcal{F}}}(V) \cong C_2\backslash\Sigma_3.$$

Thus it follows that the quiver of $\overline{\mathcal{F}}(b_0)$ is

$$\overset{V}{\bullet} \longrightarrow \overset{P}{\bullet} \longleftarrow \overset{Q}{\bullet}$$

1.4. Proof of Proposition 3.1 when $n = 3$, $q \equiv 1 \mod 4$. Let $G = \mathrm{GL}_3(q)$ where q is a prime power such that $q \equiv 1 \mod 4$. Let 2^m be the highest 2-power dividing $q - 1$, and let η be a primitive 2^mth root of unity in $\mathbb{F}_q$. Note that $m \geq 2$. Then the subgroup P of G generated by

$$x = \begin{pmatrix} \eta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \eta \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a Sylow 2-subgroup of G. We have $P = \langle x, y, t\rangle \times \langle z\rangle \cong (\mathrm{C}_{2^m} \wr \mathrm{C}_2) \times \mathrm{C}_{2^m}$. Note that $Z(P) = \langle xy, z\rangle$ and $Z_1 := Z(G) \cap P = \langle xyz\rangle$.

Let G_0 be $\mathrm{GL}_2(q)$ embedded in $\mathrm{GL}_3(q)$ in the upper left 2×2 minor. Then $P_0 = \langle x, y, t\rangle$ is a Sylow p-subgroup of G_0. Let $\mathcal{F} = \mathcal{F}_P(G)$ and $\mathcal{F}_0 = \mathcal{F}_{P_0}(G_0)$. Then the $\mathcal{F}$-centric subgroups of P are of the form $R_0 \times \langle z\rangle$ where R_0 is an $\mathcal{F}_0$-centric subgroups of G_0. Thus, they are as follows:

(1) $\langle x, y, z\rangle$
(2) $\langle xy, tx^i, z\rangle$ where $\eta^i \neq \eta^{2j}$ for any integer j
(3) $\langle xy, x^{2^i}, tx^j, z\rangle$ where $0 \leq i \leq m-1, 0 \leq j < 2^i$

Let R be an $\mathcal{F}$-centric subgroup of P. If $R = \langle x, y, z\rangle$, then we have

$$\mathrm{Aut}_{\mathcal{F}}(R) \cong N_G(R)/RC_G(R) = R\Sigma_3/R \cong \Sigma_3,$$

where Σ_3 is viewed as the subgroup of G consisting of the permutation matrices in G. If R is of type (2) or (3), then $\mathrm{Aut}_{\mathcal{F}}(R)$ fixes xy and z inducing a surjective group homomorphism

$$\mathrm{Aut}_{\mathcal{F}}(R) \twoheadrightarrow \mathrm{Aut}_{G/\langle xy,z\rangle}(R/\langle xy, z\rangle).$$

Thus the same argument as in Section 1.3 applies, and we get the desired result.

REMARK 3.2. For all the cases that we have considered in this section, $\mathrm{Aut}_{\overline{\mathcal{F}}}(Q) \cong C_2$ or Σ_3. Thus $\mathcal{A}^1 = 0$, and hence $\alpha = 0$ is the unique solution to the gluing problem. (See §5.3.)

2. The q-Schur Algebra

We review the definition and some basic properties of the q-Schur algebra defined by Dipper and James **[12]**, following the presentation of Mathas **[28]**.

Let k be a field, and let q be a nonzero element of k. The *Iwahori-Hecke algebra* of the symmetric group Σ_n on n letters is the k-algebra $\mathcal{H} = \mathcal{H}_{k,q}(\Sigma_n)$ whose k-basis is $\{T_w \mid w \in \Sigma_n\}$ and such that the multiplication is given by

$$T_wT_s = \begin{cases} T_{ws}, & \text{if } l(ws) > l(w), \\ qT_{ws} + (q-1)T_w, & \text{if } l(ws) < l(w), \end{cases}$$

where $w \in \Sigma_n$, $s = (i, i+1) \in \Sigma_n$ for some $0 < i < n$, and $l(w)$ is the length of w.

A *composition* of n is a sequence $\mu = (\mu_1, \mu_2, \dots)$ of nonnegative integers μ_i whose sum is equal to n. The *height* of a composition μ is the smallest positive integer h such that $\mu_{h+1} = \mu_{h+2} = \cdots = 0$. For a composition μ of n with height h, let Σ_μ be the corresponding *Young subgroup* of Σ_n isomorphic to $\Sigma_{\mu_1} \times \Sigma_{\mu_2} \times \cdots \times \Sigma_{\mu_h}$. Set $m_\mu = \sum_{w\in\Sigma_\mu} T_w$ and $M^\mu = m_\mu\mathcal{H}$.

DEFINITION 3.3. Let $\Lambda(n, d)$ be the set of all compositions of n with height $\leq d$. Then the *q-Schur algebra* is the endomorphism algebra

$$\mathcal{S}_{n,d}(q) = \mathrm{End}_{\mathcal{H}}\Big(\bigoplus_{\mu\in\Lambda(n,d)} M^\mu\Big).$$

We write $\mathcal{S}_n(q) = \mathcal{S}_{n,n}(q)$.

The q-Schur algebra has the following properties:

THEOREM 3.4. *Let k be a field , and let q be a nonzero element of k. Then the q-Schur algebra $\mathcal{S}_{n,d}(q)$ over k is quasi-hereditary. If* char $k = l > 0$ *and q is a prime power which is coprime to l, then the decomposition matrix of $k\mathrm{GL}_n(q)$ is completely determined by the decomposition matrices of the q^r-Schur algebras $\mathcal{S}_m(q^r)$ over k for $rm \leq n$.*

PROOF. Corollary 4.16 and Theorem 6.47 of **[28]**. □

Gruber and Hiss **[20]** and Takeuchi **[34]** give an alternative way of computing the Morita types of the q-Schur algebras. Let $G = \mathrm{GL}_n(q)$, and let B be the set of all upper triangular matrices in G.

THEOREM 3.5. *Let k be a field of characteristic $l > 0$, and let q be a prime power which is coprime to l. Then the q-Schur algebra $\mathcal{S}_n(q)$ over k is Morita equivalent to the image of the k-algebra homomorphism*

$$kG \to \mathrm{End}_k(kG/B)$$

sending $a \in kG$ to the k-linear endomorphism of kG/B given by left multiplication with a on kG/B.

3. The quivers of the q-Schur algebras of finite representation type

In this section, we summarize results of Erdmann and Nakano **[15]** **[16]** for the finite representation type case. Let k be an algebraically closed field of characteristic $l > 0$, and let q be a prime power which is coprime to l. Let e be the smallest positive integer such that

$$1 + q + q^2 + \cdots + q^{e-1} \equiv 0 \mod l.$$

Let λ be a partition of n, that is, a composition of n such that $\lambda_i \geq \lambda_{i+1}$ for all i. The *e-core* of λ is the partition whose Young diagram is obtained by removing successively as many e-rim hooks as possible from the Young diagram of λ. The *e-weight* $w(\lambda)$ of λ is the number of removals of e-rim hooks from the Young diagram of λ required to obtain the Young diagram of the e-core of λ. It is well-known fact that the e-core and e-weights do not depend on the order of removals of e-rim hooks.

Let $\mathcal{H} = \mathcal{H}_{k,q}(\Sigma_n)$. For each partition λ of n, there is a *Specht module* S^λ of $\mathcal{H}$. If λ is *e-regular*, that is, if there is no i such that $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+e} \neq 0$, then

S^λ has a unique simple quotient denoted by D^λ. Moreover, such D^λ's form a set of representatives of isomorphism classes of simple $\mathcal{H}$-modules. Blocks of $\mathcal{H}$ are parametrized by the e-cores of the partitions of n; more precisely, for partitions λ and μ of n, S^λ and S^μ are in the same block of $\mathcal{H}$ if and only if λ and μ have the same e-core.

THEOREM 3.6 ([**16**, 1.1, 3.2]). *Let B_λ be the block algebra of $\mathcal{H}_{k,q}(\Sigma_n)$ containing S^λ for a partition λ of n.*

(1) *B_λ is semisimple if and only if $w(\lambda) = 0$.*
(2) *B_λ has finite representation type if and only if $w(\lambda) \leq 1$.*

For a partition λ of n, the module M^λ has a unique submodule isomorphic to the Specht module S^λ and a unique indecomposable direct summand Y^λ containing S^λ, called the *Young module*. Let $\Lambda^+(d,n)$ be the set of all partitions of n with height $\leq d$. The algebra $\mathrm{End}_{\mathcal{H}}(\bigoplus_{\lambda\in\Lambda^+(d,n)} Y^\lambda)$ is a basic algebra for $\mathcal{S}_{d,n}(q)$. If B is a block algebra of the Iwahori-Hecke algebra $\mathcal{H}$, then let

$$\mathcal{S}_B = \mathrm{End}_{\mathcal{H}}\Big(\bigoplus_{\substack{\lambda\in\Lambda^+(d,n)\\ D^\lambda\in B}} Y^\lambda\Big).$$

The algebra $\mathcal{S}_B$ is a basic algebra for a sum of blocks of $\mathcal{S}_{d,n}(q)$.

THEOREM 3.7 ([**15**, 4.3.1]). *Let B be a block algebra of $\mathcal{H}$ of finite representation type but not semisimple. If B has m partitions, then $\mathcal{S}_B$ has the quiver*

$$\begin{tikzcd} {}^{1}\bullet \arrow[r, shift left, "\alpha_1"] & {}^{2}\bullet \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & \bullet^{3} \arrow[l, shift left, "\beta_2"] & \cdots & {}^{m-1}\bullet \arrow[r, shift left, "\alpha_{m-1}"] & \bullet^{m} \arrow[l, shift left, "\beta_{m-1}"] \end{tikzcd}$$

with relations

$$\alpha_{i+1}\alpha_i = 0, \quad \beta_i\beta_{i+1} = 0, \quad \alpha_i\beta_i = \beta_{i+1}\alpha_{i+1}, \quad \beta_1\alpha_1 = 0. \quad (1 \leq i \leq m-1)$$

THEOREM 3.8 ([**15**, 1.3]). (1) *$\mathcal{S}_n(q)$ is semisimple if and only if $n < e$.*
(2) *$\mathcal{S}_n(q)$ has finite representation type if and only if $n < 2e$.*

PROPOSITION 3.9 ([**15**, 3.3(A)]). (1) *The q-Schur algebra $\mathcal{S}_2(q)$ over k is Morita equivalent to the path algebra of the quiver*

$$\begin{tikzcd} {}^{1}\bullet \arrow[r, shift left, "\beta"] & \bullet^{2} \arrow[l, shift left, "\gamma"] \end{tikzcd}$$

with relation given by $\beta\gamma = 0$.

(2) *The q-Schur algebra $\mathcal{S}_2(q)$ over k is Morita equivalent to the path algebra of the quiver*

$$\begin{tikzcd} {}^{1}\bullet \arrow[r, shift left, "\beta"] & \bullet^{2} \arrow[l, shift left, "\gamma"] & & \bullet^{3} \end{tikzcd}$$

with relation given by $\beta\gamma = 0$.

PROOF. Note that this is a special case of (3.7). We give an elementary proof of (1).

Let B be the set of all upper triangular matrices in G. For $u \in \mathbb{F}_q$, set

$$[u] := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Also set

$$t := \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}, \qquad w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where ϵ is a generator of the multiplicative group $\mathbb{F}_q^\times$. Then we have

$$G/B = \{\, B, wB, [\epsilon^i]wB \,\}_{1 \le i \le q-1}.$$

Let

$$kG \to \mathrm{End}_k(k[G/B])$$

be the k-algebra homomorphism of Theorem 3.5 and denote its image by S. This map is the k-linear extension of the group homomorphism

$$\psi : G \to \Sigma_{G/B} \hookrightarrow \mathrm{GL}_k(k[G/B])$$

where the first homomorphism sends $g \in G$ to the permutation of G/B induced by left multiplication by g and the second inclusion sends permutations of G/B to corresponding permutation matrices. Observe that the following correspondence

$$\begin{array}{cccccc} B & wB & [\epsilon]wB & [\epsilon^2]wB & \cdots & [\epsilon^{q-1}]wB \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots & \updownarrow \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} \epsilon \\ 1 \end{bmatrix} & \begin{bmatrix} \epsilon^2 \\ 1 \end{bmatrix} & \cdots & \begin{bmatrix} \epsilon^{q-1} \\ 1 \end{bmatrix} \end{array}$$

respects the G-action on G/B by left multiplication and the natural G-action on the projective line over $\mathbb{F}_q$, where $\begin{bmatrix} u \\ v \end{bmatrix}$ denotes the image of $\begin{pmatrix} u \\ v \end{pmatrix}$ in the projective line. Denote above elements by v_1, v_2, ... ,v_{q+1}, respectively, and write $V = k[G/B] =$

$kv_1 \oplus kv_2 \oplus \cdots \oplus kv_{q+1}$. Then ψ factors through

$$\mathrm{PGL}_2(q) \cong G/Z(G) \hookrightarrow \mathrm{GL}_k(V),$$

and hence

$$S = \mathrm{Im}(k\mathrm{PGL}_2(q) \to \mathrm{End}_k(V)).$$

V is a $(q+1)$-dimensional S-module with the natural S-action. Now we find its composition series. First of all, V has an obvious 1-dimensional simple S-submodule

$$V_1 = k(v_1 + v_2 + \cdots + v_{q+1}).$$

Let us denote the elements of the quotient module V/V_1 as

$$[\lambda_1, \lambda_2, \ldots, \lambda_{q+1}] := \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_{q+1} v_{q+1} + V_1$$

with $\lambda_i \in k$. Then the $(q-1)$-dimensional S-submodule V_2 of V/V_1 given by

$$V_2 = \{\, [\lambda_1, \lambda_2, \ldots, \lambda_{q+1}] \mid \lambda_1 + \lambda_2 + \ldots + \lambda_{q+1} = 0 \,\}$$

is also simple because $\mathrm{PGL}_2(q)$ acts 3-transitively on $\{\, v_1, v_2, \ldots, v_{q+1} \,\}$.(See [**29**, Table 1]) Let W be the inverse image in V of V_2. Observe that V, W are uniserial S-modules with composition series (V_1, V_2, V_1), (V_2, V_1), respectively. In particular, both V and W are indecomposable.

It is well known that $V = k[G/B]$ is a projective S-module and that there are exactly two simple S-modules up to isomorphism. Then, since S is quasi-hereditary, it follows from the composition series of V that the standard modules for V_1 and V_2 are V_1 and W, respectively, and W is also projective. Therefore we conclude that S, and hence the q-Schur algebra $\mathcal{S}_2(q)$, is Morita equivalent to the path algebra of the given quiver with relation. □

4. Structural Connections

As a direct consequence of Propositions 3.1 and 3.9, we get the following structural relations between weighted fusion category algebras and q-Schur algebras for $\mathrm{GL}_n(q)$, $n = 2, 3$.

THEOREM 3.10. *Let k be an algebraically closed field of characteristic 2, and let q be an odd prime power. Then the weighted fusion category algebra $\overline{\mathcal{F}}(b_0)$ over k of the principal 2-block b_0 of $\mathrm{GL}_2(q)$ is Morita equivalent to the quotient of the q-Schur algebra $\mathcal{S}_2(q)$ over k by its socle.*

THEOREM 3.11. *Let k be an algebraically closed field of characteristic 2, and let q be an odd prime power. Then the basic algebra X of the weighted fusion category algebra $\overline{\mathcal{F}}(b_0)$ over k of the principal 2-block b_0 of $\mathrm{GL}_3(q)$ and the basic algebra Y of the q-Schur algebra $\mathcal{S}_3(q)$ over k are part of the following pull-back diagram*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

where Z is given by the quiver

$$\begin{tikzcd} {}^1\bullet \arrow[r] & {}^2\bullet & & \bullet^3 \end{tikzcd}$$

and W is given by the quiver where Z is given by the quiver

$$\begin{tikzcd} {}^1\bullet \arrow[r, shift left, "\beta"] & \bullet^2 \arrow[l, shift left, "\gamma"] & & \bullet^3 \arrow[ll] \end{tikzcd}$$

with relation $\beta\gamma = 0$.

5. A remark on a canonical bijection between simple modules and weights

Let k be an algebraically closed field of characteristic 2 and let q be an odd prime power. Let b_0 be the principal 2-block of $G = \mathrm{GL}_n(q)$. The algebra homomorphism in Theorem 3.5 restricts to the surjective algebra homomorphism

$$kGb_0 \twoheadrightarrow S$$

where S is a k-algebra which is Morita equivalent to the q-Schur algebra $\mathcal{S}_n(q)$. On the other hand, in Theorem 3.10 we showed that there is another surjective algebra homomorphism

$$S_0 \twoheadrightarrow T_0$$

where S_0 and T_0 are, respectively, the basic algebras of the q-Schur algebra $\mathcal{S}_n(q)$ and the weighted fusion category algebra $\overline{\mathcal{F}}(b_0)$ when $n = 2$. Combining these two surjective algebra homomorphisms, we see that simple $\overline{\mathcal{F}}(b_0)$-modules can be viewed as simple kGb_0-modules when $n = 2$. Since we have

$$l(\overline{\mathcal{F}}(b_0)) = \text{number of partitions of } n = l(kGb_0)$$

for $n = 2$ (in fact, for every n by An [**5**]), we get a canonical bijection between simple kGb_0-modules and simple $\overline{\mathcal{F}}(b_0)$-modules in this case. But there is a canonical bijection between the set of isomorphism classes of simple $\overline{\mathcal{F}}(b_0)$-modules and the set of conjugacy classes of b_0-weights. Thus we get a canonical bijection between the set of isomorphism classes of simple kGb_0-modules and the set of conjugacy classes of b_0-weights when $n = 2$.

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