COUNTING CONJUGACY CLASSES OF CYCLIC SUBGROUPS FOR FUSION SYSTEMS

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Abstract. We give another proof of an observation of Thévenaz [4] and present a fusion system version of it. Namely, for a saturated fusion system $\mathcal{F}$ on a finite p-group S, we show that the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of S is equal to the rank of certain square matrices of numbers of orbits, coming from characteristic bisets, the characteristic idempotent and finite groups realizing the fusion system $\mathcal{F}$ as in our previous work [2].

1. Statements of the results

In [4], Thévenaz observed the 'curiosity' that a finite cyclic group G can be characterized by the nonsingularity of the matrix of the numbers of double cosets in G. In fact He proved a more general fact that for an arbitrary finite group G the number of the conjugacy classes of cyclic subgroups of G is equal to the rank of that matrix. This can be stated slightly more generally by introducing a subgroup H of G and considering the G-conjugacy classes of subgroups of H as follows.

Theorem 1. *Let G be a finite group and let $H \leq G$. The rank of the matrix*

$$(|P\backslash G/Q|)_{P,Q \leq_G H},$$

whose rows and columns are indexed by the G-conjugacy classes of subgroups of H and whose entries are the numbers of the corresponding double cosets in G, is equal to the number of the G-conjugacy classes of cyclic subgroups of H.

In [2], we observed that every saturated fusion system $\mathcal{F}$ on a finite p-group S can be realized by a finite group G containing S as a (not necessarily Sylow) p-subgroup. Thus the above theorem yields a fusion system version as follows.

Theorem 2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let G be a finite group which contains S as a subgroup and realizes $\mathcal{F}$. Then the rank of the matrix*

$$(|P\backslash G/Q|)_{P,Q \leq_G S}$$

is equal to the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of S.

By a result of Broto, Levi and Oliver [1, Proposition 5.5], every saturated fusion system $\mathcal{F}$ on a finite p-group S has a (non-unique) characteristic biset Ω. See Section 3 for a precise definition; in particular, Ω is a finite (S, S)-biset, i.e., a finite set with compatible left and right S-actions. If $\mathcal{F}$ is the fusion system of a finite group G on its Sylow p-subgroup S, then G is a characteristic biset for $\mathcal{F}$ with the obvious S-action on the left and right. So we may well expect that the matrix of the above theorem with G replaced by Ω has the same rank. Indeed this is the case.

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Theorem 3. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let Ω be a characteristic biset for $\mathcal{F}$. Then the rank of the matrix*

$$(|P\backslash\Omega/Q|)_{P,Q\leq_{\mathcal{F}}S}$$

of the number of (P,Q)-orbits of Ω indexed by the $\mathcal{F}$-conjugacy classes of subgroups of S is equal to the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of S.

Finally, one can replace the characteristic biset Ω in the above theorem by the characteristic idempotent $\omega_{\mathcal{F}}$ (which is a virtual (S,S)-biset; see Section 3) with $|P\backslash\omega_{\mathcal{F}}/Q|$ as the linearized number of (P,Q)-orbits.

Theorem 4. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let $\omega_{\mathcal{F}}$ be the characteristic idempotent for $\mathcal{F}$. Then the rank of the matrix*

$$(|P\backslash\omega_{\mathcal{F}}/Q|)_{P,Q\leq_{\mathcal{F}}S},$$

is equal to the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of S.

We will give a proof of Theorem 1 (and hence obtain Theorem 2 as a corollary), which is slightly different from that of [4]. This new proof uses (at least explicitly) only the Burnside ring $B(G)$ of G, not the rational representation ring $R_{\mathbb{Q}}(G)$ as in [4]. Therefore it is better suited for adapting to the fusion system case (Theorem 3 and 4), which we do subsequently.

2. The group case

We prove Theorem 1. As remarked in Section 1, Theorem 2 then immediately follows as a corollary.

Let G be a finite group. Let $B(G)$ be the Burnside ring of G, i.e., the Grothendieck ring of the isomorphism classes $[X]$ of finite G-sets X. As an additive group, $B(G)$ is a free abelian group with the canonical basis $\{[G/P] \mid P \leq_G G\}$. Let $\mathbb{Q}B(G) = \mathbb{Q} \otimes_{\mathbb{Z}} B(G)$ and regard $B(G)$ as a subring of $\mathbb{Q}B(G)$. In particular the canonical basis for $B(G)$ is a $\mathbb{Q}$-basis for $\mathbb{Q}B(G)$.

It is a well-known fact that for each $P \leq G$ the fixed-point map

$$\chi_P \colon B(G) \to \mathbb{Z}, \quad [X] \mapsto |X^P|,$$

is a ring homomorphism which depends only on the G-conjugacy class of P, and their product (tensored with $\mathbb{Q}$)

$$\chi = \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{P\leq_G G} \chi_P \colon \mathbb{Q}B(G) \to \prod_{P\leq_G G} \mathbb{Q}$$

is a $\mathbb{Q}$-algebra isomorphism. For each $P \leq G$, let e_P^G denote the element of $\mathbb{Q}B(G)$ such that

$$\chi_Q(e_P^G) = \begin{cases} 1, & P =_G Q, \\ 0, & \text{otherwise.} \end{cases}$$

Then again the element e_P^G depends only on the G-conjugacy class of P and $\{e_P^G \mid P \leq_G G\}$ is a set of pairwise orthogonal primitive idempotents of $\mathbb{Q}B(G)$ whose sum is equal to 1; in particular it is a $\mathbb{Q}$-basis for $\mathbb{Q}B(G)$. Furthermore, for $H \leq G$, let $B(G)_H$ be the subgroup of $B(G)$ generated by the elements $[G/P]$ with $P \leq_G H$. Then $\mathbb{Q}B(G)_H = \mathbb{Q} \otimes_{\mathbb{Z}} B(G)_H$ is a subalgebra of $\mathbb{Q}B(G)$ with $\mathbb{Q}$-basis $\{[G/P] \mid P \leq_G H\}$. Note that the elements e_P^G with $P \leq_G H$ belong to $\mathbb{Q}B(G)_H$ and hence $\{e_P^G \mid P \leq_G H\}$ is another basis for $\mathbb{Q}B(G)_H$.

For each $P \leq G$ consider the $\mathbb{Q}$-linear map

$$\rho_P \colon \mathbb{Q}B(G) \to \mathbb{Q}, \quad [X] \mapsto |P\backslash X|,$$

which counts the P-orbits. By Burnside's orbit counting lemma, we have

$$\rho_P(x) = \frac{1}{|P|} \sum_{u \in P} \chi_{\langle u \rangle}(x), \quad x \in \mathbb{Q}B(G).$$

Thus

$$\rho_P(e_Q^G) \neq 0 \iff Q \text{ is cyclic and } Q \leq_G P. \tag{1}$$

Now the given matrix in Theorem 1 is equal to

$$(\rho_P(G/Q))_{P,Q \leq_G H}.$$

By change of basis, this matrix has the same rank as

$$(\rho_P(e_Q^G))_{P,Q \leq_G H}.$$

List the subgroups of H (up to G-conjugacy) in two blocks, the first consisting of cyclic subgroups and the second of noncyclic subgroups, and with nondecreasing order in each block. Then by (1) the above matrix has the form

$$\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

where A is a lower triangular matrix with nonzero diagonal entries. Thus Theorem 1 follows.

3. The fusion system case

We first prove Theorem 3. In fact, we prove a slightly generalized version of it.

Proposition 5. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Suppose that Ω is a finite (S,S)-biset which is $\mathcal{F}$-stable and $\mathcal{F}$-generated and which contains the obvious (S,S)-biset S. Then the rank of the matrix*

$$(|P\backslash\Omega/Q|)_{P,Q \leq_{\mathcal{F}} S}$$

is equal to the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of S.

We first explain the terminology. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. An S-set X is *$\mathcal{F}$-stable* if, for all $P \leq S$ and all $\mathcal{F}$-morphism $\varphi \colon P \to S$, the restrictions of the S-action on X to P via the inclusion $P \hookrightarrow S$ and via $\varphi \colon P \to S$ give isomorphic P-sets. We say that an (S,S)-biset is *$\mathcal{F}$-stable* if it is $\mathcal{F} \times \mathcal{F}$-stable viewed as a left $S \times S$-set by inverting the right action of S. An (S,S)-biset is *$\mathcal{F}$-generated* if, viewed as a left $S \times S$-set, all its isotropy subgroups are of the form $\Delta(P,\varphi) = \{(u,\varphi(u)) \mid u \in P\}$ with $P \leq S$, $\varphi \colon P \to S$ in $\mathcal{F}$. A finite (S,S)-biset Ω is called a *characteristic biset* for $\mathcal{F}$ if it is $\mathcal{F}$-stable and $\mathcal{F}$-generated and such that $|\Omega|/|S|$ is not divisible by p. It is easy to see that every characteristic biset Ω contains the (S,S)-biset S.

Define

$$B(\mathcal{F}) = \{x \in B(S) \mid \chi_P(x) = \chi_{P'}(x) \text{ for all } P, P' \leq S \text{ with } P =_{\mathcal{F}} P'\}.$$

Clearly $B(\mathcal{F})$ is a subring of $B(S)$, which is called the Burnside ring of the fusion system $\mathcal{F}$. For a finite S-set X, we have $[X] \in B(\mathcal{F})$ if and only if X is $\mathcal{F}$-stable. As before let $\mathbb{Q}B(\mathcal{F}) = \mathbb{Q} \otimes_{\mathbb{Z}} B(\mathcal{F})$. Clearly the elements

$$e_P^{\mathcal{F}} := \sum_{P' =_{\mathcal{F}} P} e_{P'}^S,$$

where $P \leq S$ and the sum is over the S-conjugacy classes of subgroups P' of S which are $\mathcal{F}$-conjugate to P, belong to $\mathbb{Q}B(\mathcal{F})$. The set $\{e_P^{\mathcal{F}} \mid P \leq_{\mathcal{F}} S\}$ is a set of pairwise orthogonal primitive idempotents of $\mathbb{Q}B(\mathcal{F})$ whose sum is equal to 1; in particular it is a $\mathbb{Q}$-basis for $\mathbb{Q}B(\mathcal{F})$. By (1), we have

$$\rho_P(e_Q^{\mathcal{F}}) \neq 0 \iff Q \text{ is cyclic and } Q \leq_{\mathcal{F}} P. \tag{2}$$

Let Ω be the (S,S)-biset given in the above proposition. By the $\mathcal{F}$-stability of Ω, the left S-set Ω/P of the right P-orbits of Ω is also $\mathcal{F}$-stable for $P \leq S$. Moreover

$$\chi_Q([\Omega/P]) \neq 0 \implies Q \leq_{\mathcal{F}} P; \quad \chi_P([\Omega/P]) \geq |N_S(P)/P|.$$

The former follows from that Ω is $\mathcal{F}$-generated and the latter from that Ω contains S. Hence

$$\{[\Omega/P] \mid P \leq_{\mathcal{F}} S\}$$

is a $\mathbb{Q}$-basis for $\mathbb{Q}B(\mathcal{F})$. Thus the matrix

$$(|P\backslash\Omega/Q|)_{P,Q \leq_{\mathcal{F}} S} = (\rho_P([\Omega/Q]))_{P,Q \leq_{\mathcal{F}} S}$$

has the same rank as

$$(\rho_P(e_Q^{\mathcal{F}}))_{P,Q \leq_{\mathcal{F}} S},$$

which is equal to the number of the $\mathcal{F}$-conjugacy classes of cyclic subgroups of S by (2).

Remark. Note that the finite group G in Theorem 2, viewed as an (S,S)-biset, satisfies the hypotheses for Ω in Proposition 5. Thus Theorem 2 can also be obtained from Proposition 5.

Now we address Theorem 4. In Proposition 5, the condition that Ω contains the (S,S)-biset S is equivalent to that $\chi_P(\Omega/P) \neq 0$ for all $P \leq S$, given the other conditions on Ω. Proposition 5 then applies to all virtual (S,S)-bisets ω with coefficients in $\mathbb{Q}$ which are $\mathcal{F}$-stable, $\mathcal{F}$-generated and such that $\chi_P(\omega/P) \neq 0$ for all $P \leq S$, where ω/P denotes the linearized right P-orbits of ω. The proof is identical to the one given above. In particular, Reeh [3, Proposition 4.5, Corollary 5.8] shows that if ω is the *characteristic idempotent* of $\mathcal{F}$, i.e., the unique virtual (S,S)-biset with coefficients in $\mathbb{Z}_{(p)}$ which is $\mathcal{F}$-stable, $\mathcal{F}$-generated and which is an idempotent in the double Burnside ring $\mathbb{Z}_{(p)}B(S,S)$, then the elements $\omega/P = \omega \circ_S [S/P] = \beta_P$ with $P \leq_{\mathcal{F}} S$ form a basis of $\mathbb{Z}_{(p)}B(\mathcal{F})$ such that $\chi_P(\omega/P) \neq 0$. This proves Theorem 4.

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