Newton’s method can be considered to be a special case of a very general approach called *Fixed Point Iteration* or *SimpleIteration*.

The basic idea is:

*If we want to solve $f(x) = 0$ in $[a, b]$, find a function $g(x)$ such that, if $\tau$ is such that $f(\tau) = 0$, then $g(\tau) = \tau$. Choose $x_0$ and set $x_{k+1} = g(x_k)$ for $k = 0, 1, 2, \ldots$.***
Example 1.11

Suppose that \( f(x) = e^x - 2x - 1 \) and we are trying to find a solution to \( f(x) = 0 \) in \([1, 2]\). Then we can take \( g(x) = \ln(2x + 1) \).

If we take \( x_0 = 1 \), then we get the following sequence:

| \( k \) | \( x_k \) | \( |r - x_k| \) |
|---|---|---|
| 0 | 1.0000 | 2.564e-1 |
| 1 | 1.0986 | 1.578e-1 |
| 2 | 1.1623 | 9.415e-2 |
| 3 | 1.2013 | 5.509e-2 |
| 4 | 1.2246 | 3.187e-2 |
| 5 | 1.2381 | 1.831e-2 |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |
| 10 | 1.2558 | 6.310e-4 |
We have to be quite careful with this method: **not every choice is $g$ is suitable.**

For example, suppose we want the solution to $f(x) = x^2 - 2 = 0$ in $[1, 2]$. We could choose $g(x) = x^2 + x - 2$. Then, if take $x_0 = 1$ we get the sequence:

We need to refine the method that ensure that it will converge.
Before we do that in a formal way, consider the following...

Example 1.12

Use the Mean Value Theorem to show that the fixed point method \( x_{k+1} = g(x_k) \) converges if \( |g'(x)| < 1 \) for all \( x \) near the fixed point.

This example:

- introduces the tricks of using that \( g(\tau) = \tau \) & \( g(x_k) = x_{k+1} \).
- Leads us towards the \textit{contraction mapping theorem}. 
Theorem 1.13 (Fixed Point Theorem)

Suppose that $g(x)$ is defined and continuous on $[a, b]$, and that $g(x) \in [a, b]$ for all $x \in [a, b]$. Then there exists $\tau \in [a, b]$ such that $g(\tau) = \tau$. That is, $g(x)$ has a fixed point in $[a, b]$. 

![Graph of a function $g(x)$ with fixed point $\tau$ within the interval $[a, b]$.]
Next suppose that $g$ is a contraction. That is, $g(x)$ is continuous and defined on $[a, b]$ and there is a number $L \in (0, 1)$ such that

$$|g(\alpha) - g(\beta)| \leq L|\alpha - \beta| \text{ for all } \alpha, \beta \in [a, b].$$

(8)

\textbf{Theorem 1.14 (Contraction Mapping Theorem)}

Suppose that the function $g$ is a real-valued, defined, continuous, and

(a) maps every point in $[a, b]$ to some point in $[a, b]$, and

(b) is a contraction on $[a, b]$,

then

(i) $g(x)$ has a fixed point $\tau \in [a, b]$,

(ii) the fixed point is unique,

(iii) the sequence $\{x_k\}_{k=0}^{\infty}$ defined by $x_0 \in [a, b]$ and $x_k = g(x_{k-1})$ for $k = 1, 2, \ldots$ converges to $\tau$. 
The algorithm generates as sequence \( \{x_0, x_1, \ldots, x_k\} \). Eventually we must stop. Suppose we want the solution to be accurate to say \( 10^{-6} \), how many steps are needed? That is, how big do we need to take \( k \) so that

\[ |x_k - \tau| \leq 10^{-6}? \]

The answer is obtained by first showing that

\[ |\tau - x_k| \leq \frac{L^k}{1 - L}|x_1 - x_0|. \]  

(9)
Example 1.15

Suppose we are using FPI to find the fixed point $\tau \in [1, 2]$ of $g(x) = \ln(2x + 1)$ with $x_0 = 1$, and we want $|x_k - \tau| \leq 10^{-6}$, then we can use (9) to determine the number of iterations required.
Exercise 1.14
Is it possible for $g$ to be a contraction on $[a, b]$ but not have a fixed point in $[a, b]$? Give an example to support your answer.

Exercise 1.15 (⋆ Homework problem)
Show that $g(x) = \ln(2x + 1)$ is a contraction on $[1, 2]$. Give an estimate for $L$. (Hint: Use the Mean Value Theorem).
Exercises

Exercise 1.16

Suppose we wish to numerically estimate the famous golden ratio, \( \tau = \frac{1 + \sqrt{5}}{2} \), which is the positive solution to \( x^2 - x - 1 \). We could attempt to do this by applying fixed point iteration to the functions \( g_1(x) = x^2 - 1 \) or \( g_2(x) = 1 + \frac{1}{x} \) on the region \([3/2, 2]\).

(i) Show that \( g_1 \) is not a contraction on \([3/2, 2]\).

(ii) Show that \( g_2 \) is a contraction on \([3/2, 2]\), and give an upper bound for \( L \).

Exercise 1.17

Consider the function \( g(x) = \frac{x^2}{4} + \frac{5x}{4} - \frac{1}{2} \).

(i) It has two fixed points – what are they?

(ii) For each of these, find the largest region around them such that \( g \) is a contraction on that region.
(i) Prove that if \( g(\tau) = \tau \), and the fixed point method given by

\[
x_{k+1} = g(x_k),
\]

converges to the point \( \tau \) (where \( g(\tau) = \tau \)), and

\[
g'(\tau) = g''(\tau) = \cdots = g^{(p-1)}(\tau) = 0,
\]

then it converges with order \( p \). (Hint: you don’t have to prove that the method converges; you can assume that. Also, use a Taylor Series).

(ii) We can think of Newton’s Method for the problem \( f(x) = 0 \) as fixed point iteration with \( g(x) = x - f(x)/f'(x) \). Use this, and Part (i), to show that, if Newton’s method converges, it does so with order 2, providing that \( f'(\tau) \neq 0 \).