MA284 : Discrete Mathematics

Week 11: Eulerian and Hamiltonian graphs; Trees

http://www.maths.nuigalway.ie/~niall/MA284/

20 and 22 November, 2019

1. Recall: Eulerian Paths and Circuits
2. Hamiltonian Paths and Cycles
3. Trees
   - A property of trees
   - Recognising trees from quite a long way away
   - Applications: Chemistry
4. Exercises

See also Chapter 4 of Levin’s *Discrete Mathematics: an open introduction.*
ASSIGNMENT 4 has finished, and results have been posted to Blackboard.

ASSIGNMENT 5 is Open. Deadline 5pm, 29 November.

Access it at
http://mathswork.nuigalway.ie/webwork2/1920-MA284

For more information, see Blackboard, or
http://www.maths.nuigalway.ie/~niall/MA284
Last week we introduced the following two ideas.

**Eulerian Path and Circuit**

An **EULERIAN PATH** (also called an *Euler Path* and an *Eulerian trail*) in a graph is a path which uses every edge exactly once.

An **EULERIAN CIRCUIT** (also called an *Eulerian cycle*) in a graph is an *Eulerian* path that starts and finishes at the same vertex.

If a graph has such a circuit, we say it is *Eulerian*.

**Example:** Find an Euler Circuit in the following graph (W3).
In the previous example, we noticed that for every edge in the circuit that “exits” a vertex, there is another that “enters” that vertex. So every vertex must have even degree.

In fact...

A graph has an **EULERIAN CIRCUIT** if and only if every vertex has even degree.

**Example:** Show that the following graph has an *Eulerian circuit*
Next suppose that a graph does not have an Eulerian circuit, but does have an Eulerian Path. Then the degree of the “start” and “end” vertices must be odd, and every other vertex has an even degree.

Example:
To summarise:

**Eulerian Paths and Circuits**

- A graph has an **EULERIAN CIRCUIT if and only if** the degree of every vertex is even.
- A graph has an **EULERIAN PATH if and only if** it has either zero or two vertices with odd degree.

**Example:** The *Königsberg bridge* graph does not have an Eulerian path:
Example (MA284, 2017/18 Semester 1 Exam)

Determine whether or not the following graph has an Eulerian Path and/or Eulerian circuit. If so, give an example; if not, explain why.
Closely related to the idea of finding path in a graph that uses every edges ones and only once, we have the following idea:

**Hamiltonian Path**

A path in a graph that visits every vertex exactly once is called a **HAMILTONIAN PATH**.

These paths are named after William Rowan Hamilton, the Irish mathematician, who invented a board-game based on the idea.

Hamilton’s Icosian Game (Library of the Royal Irish Academy)
Hamiltonian Paths and Cycles

**Hamiltonian Cycles**

Recall that a **CYCLE** is a path that starts and finishes at the same vertex, but no other vertex is repeated.

A **HAMILTONIAN CYCLE** is a cycle which visits the start/end vertex twice, and every other vertex exactly once.

A graph that has a Hamiltonian cycle is called a **HAMILTONIAN GRAPH**.

Examples:
This is the graph based on Hamilton’s Icosian game. We’ll find a Hamilton path. Can you find a Hamilton cycle?
Important examples of Hamiltonian Graphs include:

- cycle graphs;
- complete graphs;
- graphs of the platonic solids.
In general, the problem of finding a Hamiltonian path or cycle in a large graph is hard (it is known to be NP-complete). However, there are two relatively simple sufficient conditions to testing if a graph is Hamiltonian.

1. **Ore’s Theorem**

A graph with \( v \) vertices, where \( v \geq 3 \), is Hamiltonian if, for every pair of non-adjacent vertices, the sum of their degrees \( \geq v \).
2. **Dirac’s Theorem**

A (simple) graph with $v$ vertices, where $v \geq 3$, is *Hamiltonian* if every vertex has degree $\geq v/2$.

**Example**

Determine whether or not the graph illustrated below is Hamiltonian, and if so, give a Hamiltonian cycle:
Trees

There's an important class of the graph that do not contain circuits: **TREES**. The mathematical study of trees dates to at least 1857, when Arthur Cayley used them to study certain chemical compounds.

They are used in many mathematical models of decision making (such as Chess programmes), and in designing algorithms for data encoding and transmission.

**Acyclic/Forest**

A graph that has no circuits is called **ACYCLIC** or a “forest”.

**Tree**

A **TREE** is a connected, acyclic graph.
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**Examples:**
Which of the following are graphs of trees?
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If $T$ is a tree, then $e = v - 1$

If $T$ is a tree (i.e., a connected acyclic graph) with $v$ vertices, then it has $v - 1$ edges. (We will see that the converse of this statement is also true).

(See also Prop 4.2.4 in the textbook).
For a very large graph, it can be difficult to determine if it is a tree just by inspection. If we know it has no cycles, then we need to verify that it is connected. The following result (the converse of the previous one) can be useful.

**If** \( e = v - 1 \), **then** \( T \) **is a tree**

If graph with \( v \) vertices has *no* cycles, and has \( e = v - 1 \) edges, then it is a tree.
Example

The following graph has no cycles. Determine how many components it has.
Is it a tree?
There are many, many applications, of trees in mathematics, computer science, and the applied sciences. As already mentioned, the mathematical study of trees began in Chemistry.

Example

*Saturated hydrocarbons* isomers (alkane) are of the form $C_nH_{2n+2}$. They have $n$ carbon atoms, and $2n + 2$ hydrogen atoms. The carbon atoms can bond with 4 other atoms, and the hydrogens with just one. Show that the graph of all such isomers are trees.
Q1. Give an example, with justification, of an Eulerian graph that is not Hamiltonian.

Q2. For each of the following graphs, determine if it has an Eulerian path and/or circuit. If not, explain why; otherwise give an example.
   
   (a) $K_n$, with $n$ even.
   
   (b) $G_1 = (V_1, E_1)$ with $V_1 = \{a, b, c, d, e, f\}$,
       
       $E_1 = \{\{a, b\}, \{a, f\}, \{c, b\}, \{e, b\}, \{c, e\}, \{d, c\}, \{d, e\}, \{b, f\}\}$.
   
   (c) $G_2 = (V_2, E_2)$ with $V_2 = \{a, b, c, d, e, f\}$, $E_2 = \{\{a, b\}, \{a, f\}, \{c, b\}, \{e, b\}, \{c, e\}, \{d, c\}, \{d, e\}, \{b, f\}, \{b, d\}\}$.

Q3. (a) Find a Hamiltonian path in each of the graphs in Q2.
   
   (b) Determine if any of the graphs in Q2 have a Hamiltonian cycle. If not, find an induced sub-graph that does have a Hamilton cycle.

Q4. Show that $K_{3,3}$ has Hamiltonian, but $K_{2,3}$ is not.
Q5. Determine if the follow graph has an *Eulerian path* and/or *Eulerian circuit*. If so, give an example; if not, explain why.

\[ G_1 = \]

\[ G_2 = \]

Q6. (See Exer 1 in §4.2 of text). Which of the following graphs are trees?

(a) \( G = (V, E) \) with \( V = \{a, b, c, d, e\} \) and
\[
E = \{\{a, b\}, \{a, e\}, \{b, c\}, \{c, d\}, \{d, e\}\}.
\]

(b) \( G = (V, E) \), with \( V = \{a, b, c, d, e\} \) and
\[
E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}\}.
\]

(c) \( G = (V, E) \) with \( V = \{a, b, c, d, e\} \) and
\[
E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}\}.
\]

(d) \( G = (V, E) \) with \( V = \{a, b, c, d, e\} \) and \( E = \{\{a, b\}, \{a, c\}, \{d, e\}\} \).
Q7. (See Q2 in Section 4.2 of text-book). For each degree sequence below, decide whether it must always, must never, or could possibly be a degree sequence for a tree. Remember, a degree sequence lists out the degrees (number of edges incident to the vertex) of all the vertices in a graph in non-increasing order.

(a) (4, 1, 1, 1, 1)
(b) (3, 3, 2, 1, 1)
(c) (2, 2, 2, 1, 1)
(d) (4, 4, 3, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1).