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Noise in Nonsmooth Dynamical Systems

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PHD VIVA EXAMINATION
22ND NOVEMBER 2019

Acknowledgements

Outline

Background

Noise and Nonsmoothness in Dynamical Systems

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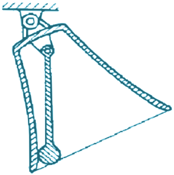
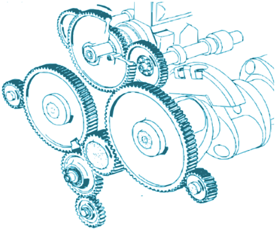
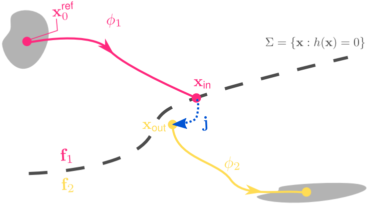
Nonsmooth Dynamical Systems:







Noise and Nonsmoothness in Dynamical Systems

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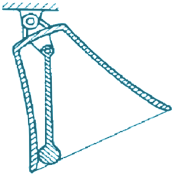

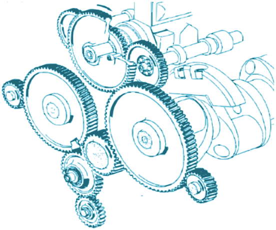

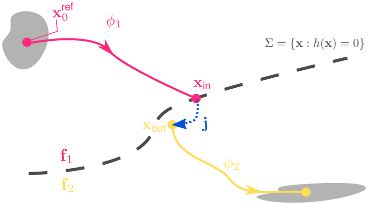

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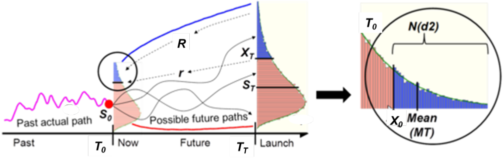

Nonsmooth Systems

Nonsmooth Maps:

A nonsmooth map is defined by a finite set of smooth maps

$$\mathbf{x}_{k+1} = \mathbf{f}_i(\mathbf{x}_k), \quad \mathbf{x} \in \mathcal{S}_i,$$

where $\cup_i \mathcal{S}_i = \mathcal{S} \subset \mathbb{R}^n$. $\Sigma_{ij} = \mathcal{S}_i \cap \mathcal{S}_j$ is either the empty set or an $(n-1)$-dimensional manifold which is the boundary between $\mathcal{S}_i$ and $\mathcal{S}_j$.

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Hybrid Systems:

A hybrid dynamical system is defined by a finite set of smooth ODEs

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plus the set of jump maps

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Discontinuity Boundaries:

We usually describe discontinuity boundaries Σ_{ij} by the zeros of a scalar function $h : \mathcal{S} \to \mathbb{R}$

$$\Sigma_{ij} = \{\mathbf{x} : h(\mathbf{x}) = 0\}.$$

The Effects of Noise and Nonsmoothness

Both noise and nonsmoothness have been shown to independently be the drivers of significant changes in behaviour.

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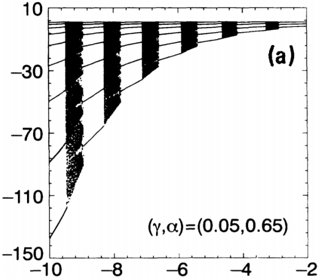

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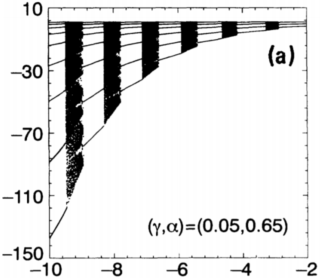

Figure: From [1]

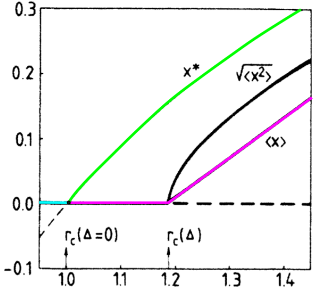

Figure: Adapted from [2]

The Square Root Map

E. J. Staunton and P. T. Piiroinen, "Noise and multistability in the square root map," *Physica D: Nonlinear Phenomena*, vol. 380, pp. 31–44, 2018.

E. J. Staunton and P. T. Piiroinen, "Noise-induced multistability in the square root map," *Nonlinear Dynamics*, vol. 95, no. 1, pp. 769–782, 2019.

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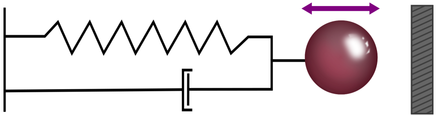

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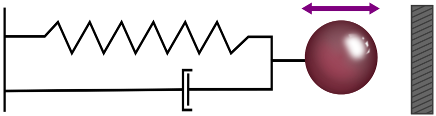

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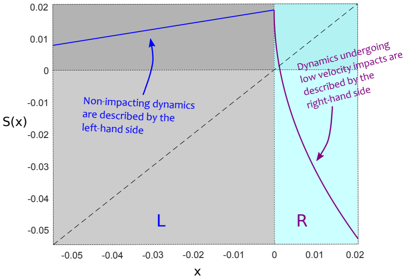

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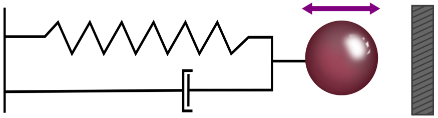

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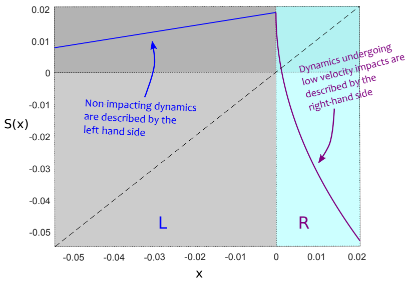

Symbolically, if $x_n < 0$ it is represented by an L and if $x_n > 0$ it is represented by an R.

The Period Adding Cascade

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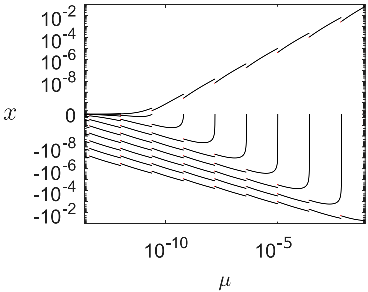

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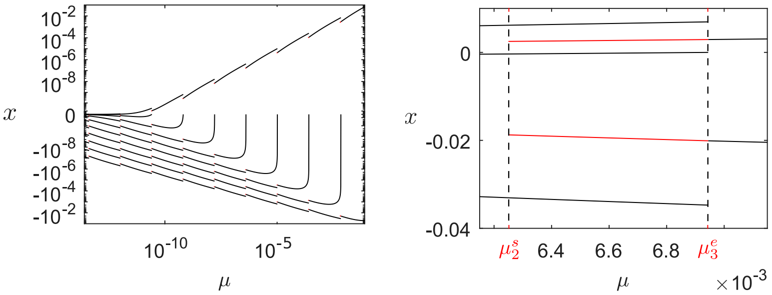

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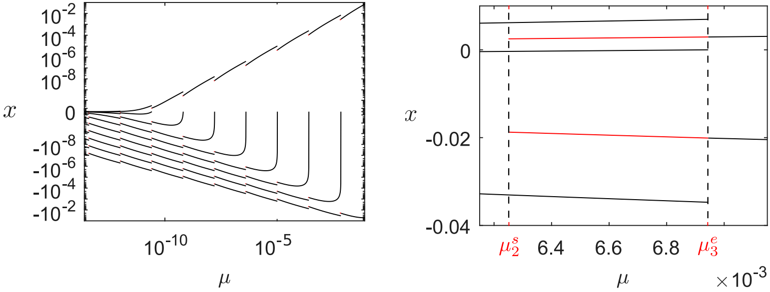

Figure: Period-adding cascade in the square root map.

These periodic orbits take the form $(RL^m)^\infty$ for $m = 1, 2, 3, \ldots$. They consist of one iterate on the right followed by m iterates on the left.

Riddled Basins of Attraction

On regions of multistability the basins of attraction of the two periodic attractors have a complex *riddled* structure.

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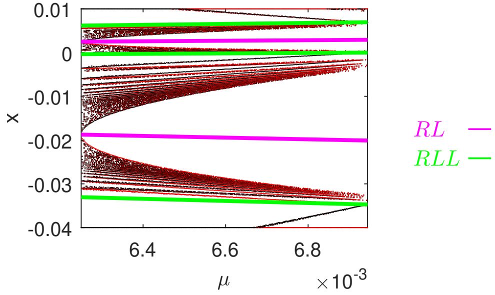

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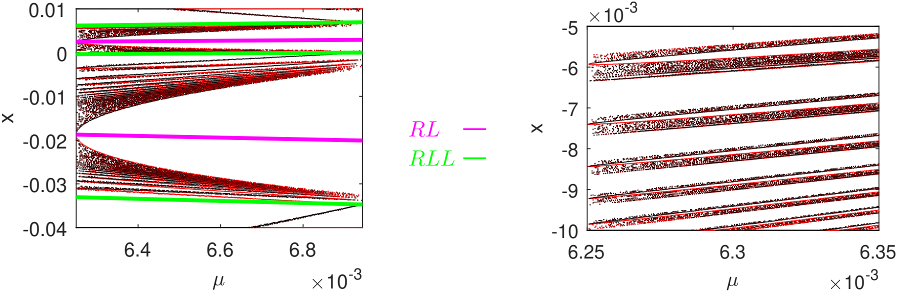

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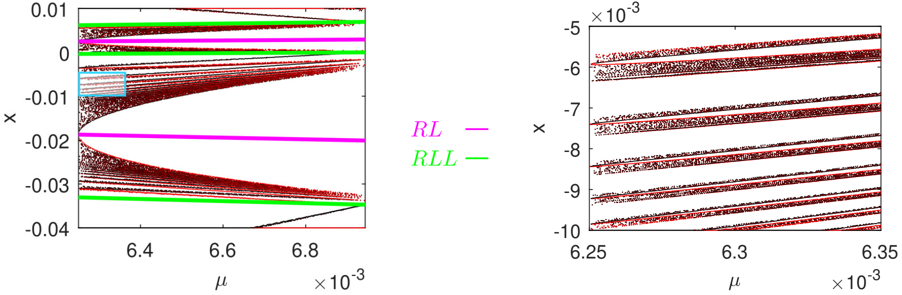

The Square Root Map With Additive Noise

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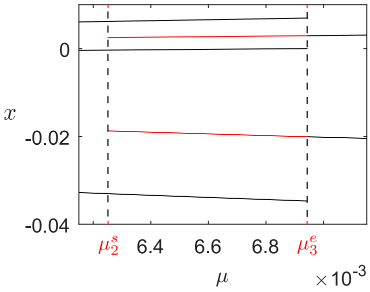
The 1-D square root map with additive Gaussian white noise is given by

$$
\begin{aligned}
x_{n+1} = S_a(x_n) &= \begin{cases} \mu + b x_n + \xi_n & \text{if } x_n < 0 \\ \mu - a\sqrt{x_n} + \xi_n & \text{if } x_n \geq 0, \end{cases} \\
\xi_n &\sim N(0, \Delta^2),
\end{aligned}
$$

where ξ_n are identically distributed independent normal random variables with mean 0 and standard deviation Δ.

Noisy Bifurcation Diagrams



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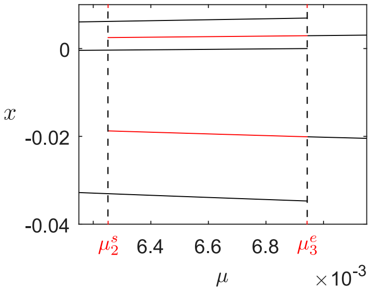

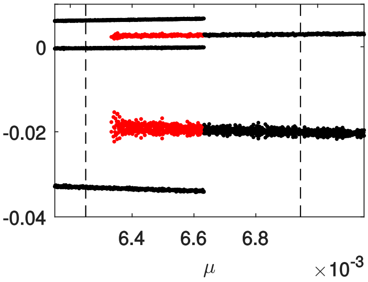

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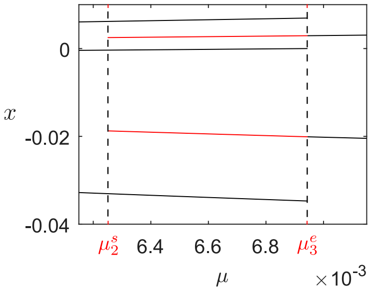
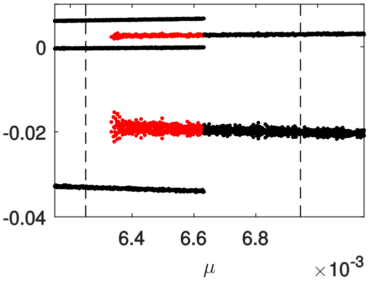
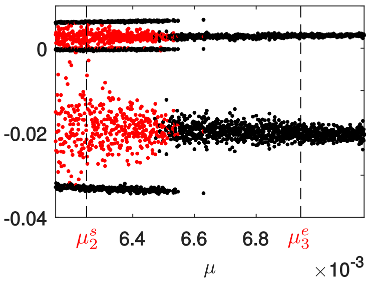
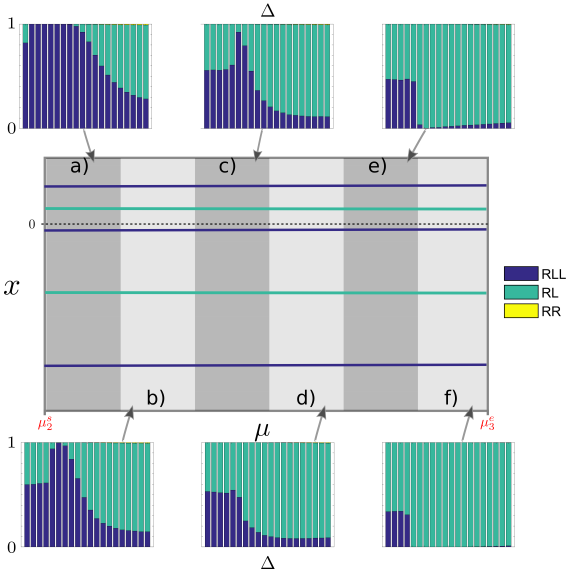






Noise Amplitude and Proportions of Periodic Behaviour



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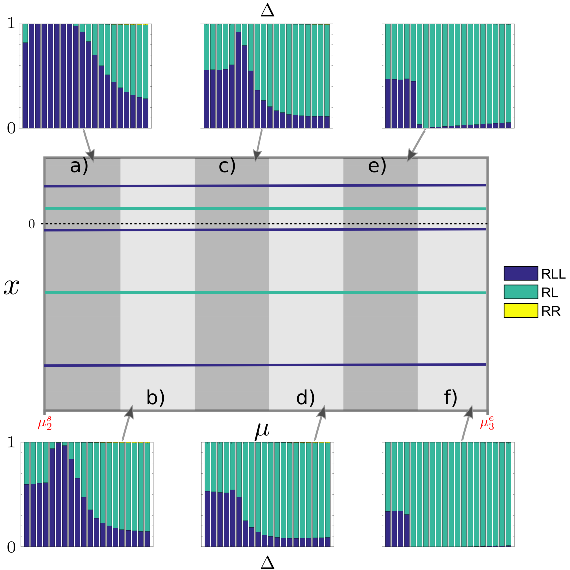

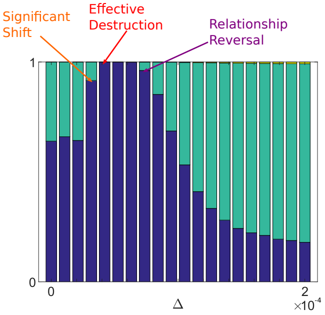

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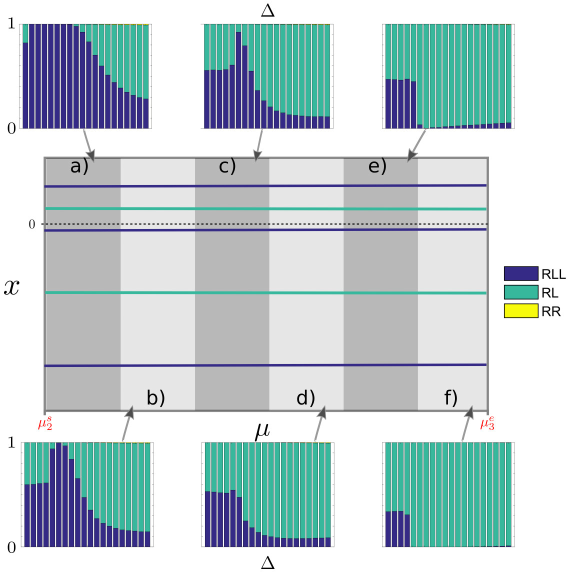
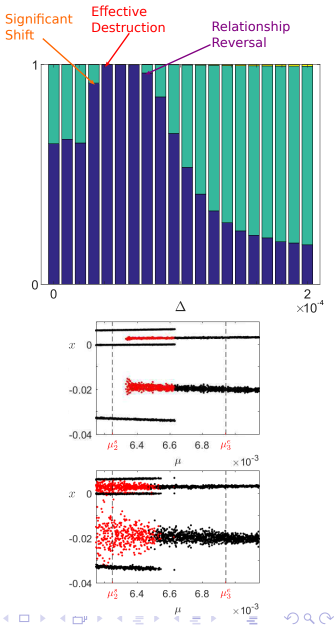
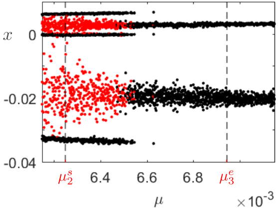




Inducing Multistability

Noise of an appropriate amplitude also has the potential to induce multistability in regions outside deterministic intervals of multistability.



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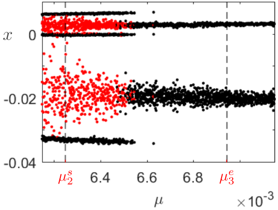

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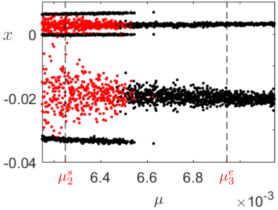

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The significant feature of the symbolic representation of the transition above is the repeated R, corresponding to repeated iteration on the right-hand side of the square root map.

Noise and Deterministic Structures

The set of initial values that are on the right which remain on the right after iteration by the deterministic square root map are given by the interval

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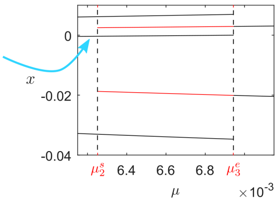
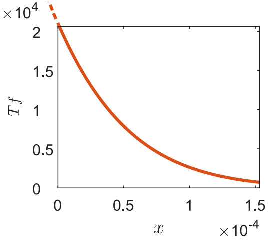
We also note that the last left iterate of the period-3 orbit is very close to 0 for values of μ close to the interval of multistability.



It is not hard to see that noise has the potential to push the last left iterate of a period-3 orbit into A_{RR} inducing repeated R's.

Noise and Deterministic Structures

RL —
RLL —



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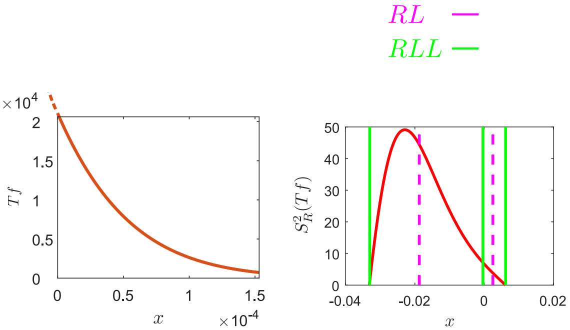

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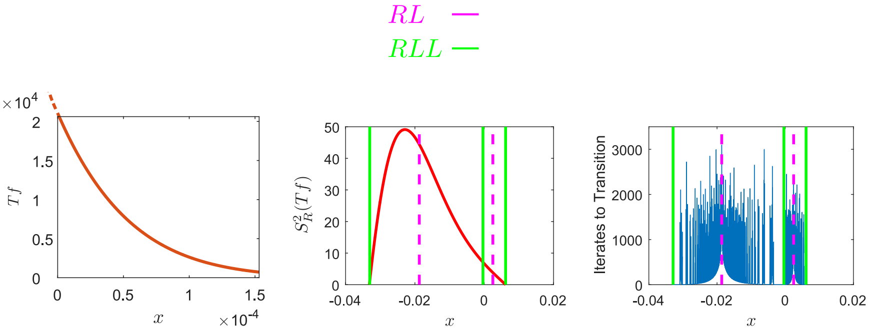

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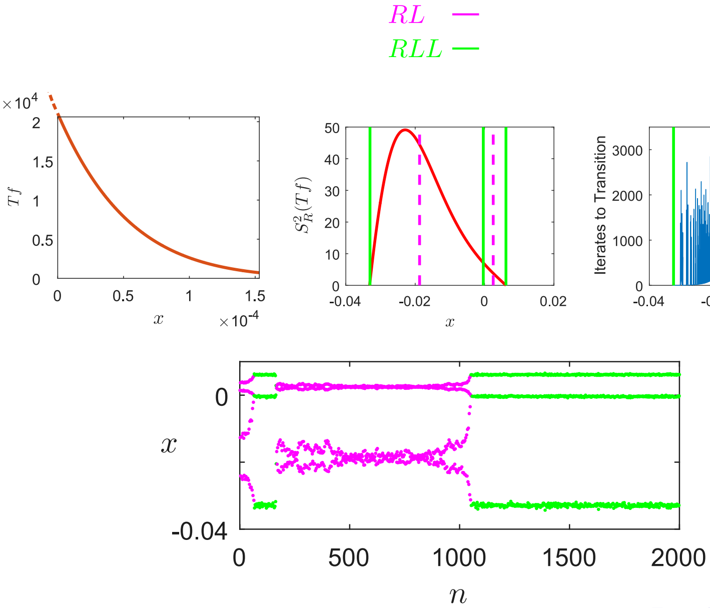

Discontinuity Mappings

E. J. Staunton and P. T. Piiroinen, "Estimating the dynamics of systems with noisy boundaries," Accepted for publication in *Nonlinear Analysis: Hybrid Systems*, 2019.

E. J. Staunton and P. T. Piiroinen, "Discontinuity mapping for stochastic nonsmooth systems," *In submission*, 2019.

Motivation

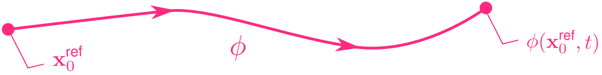

Figure: Smooth linearisation.

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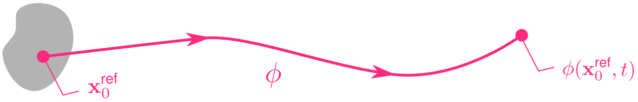

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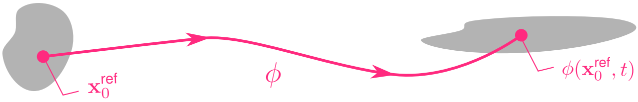

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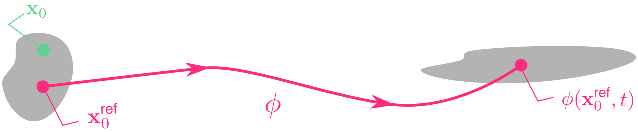

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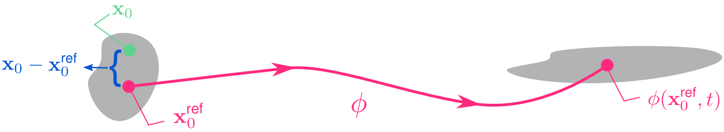

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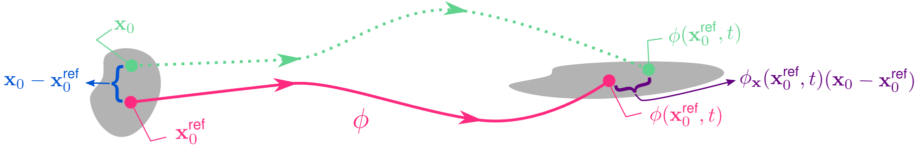

Figure: Smooth linearisation.

Transversal Crossings

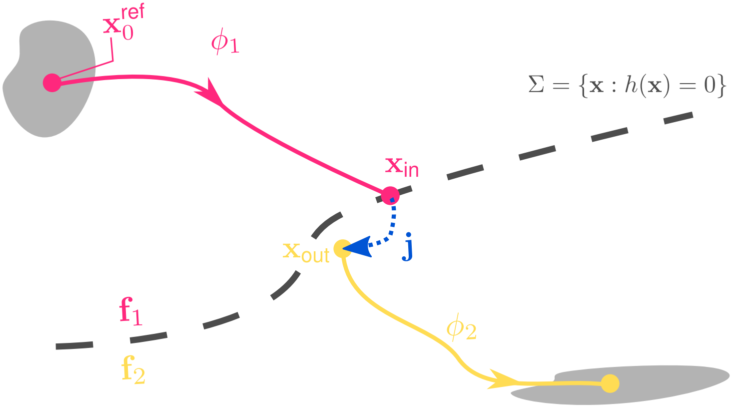

Figure: A nonsmooth dynamical system.

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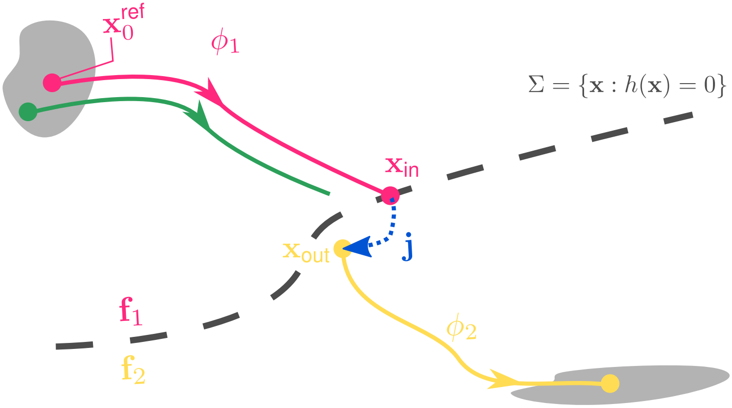

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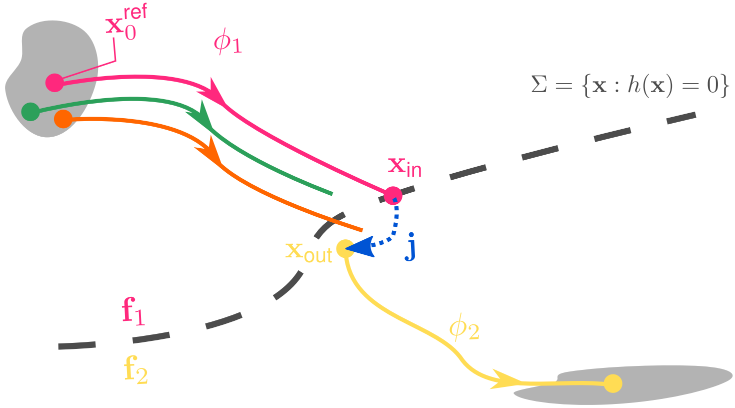

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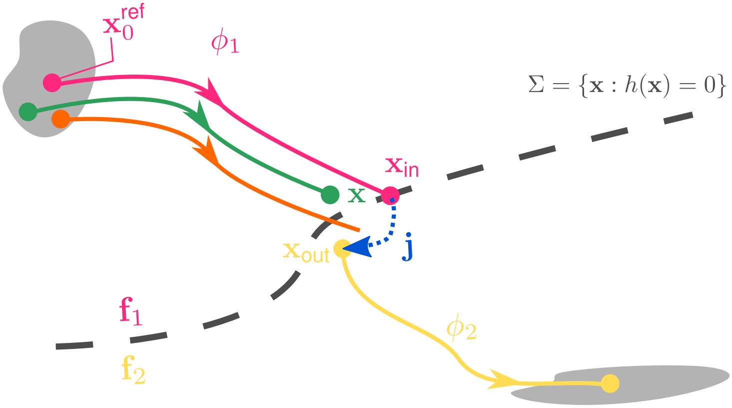

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Constructing the Zero-Time Discontinuity Mapping

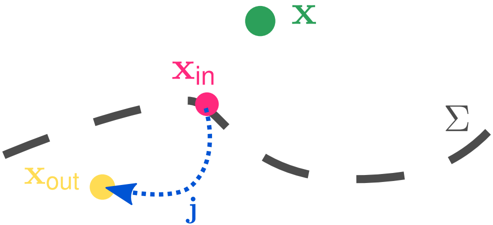

Figure: Constructing the ZDM.

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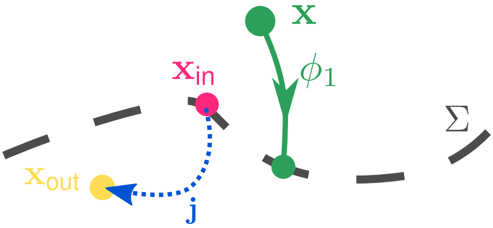

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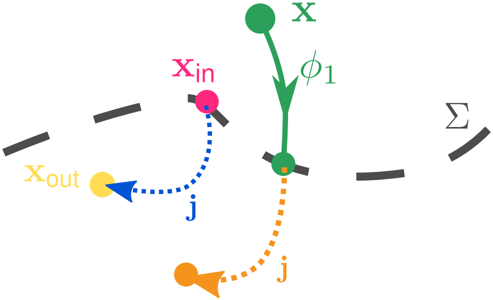

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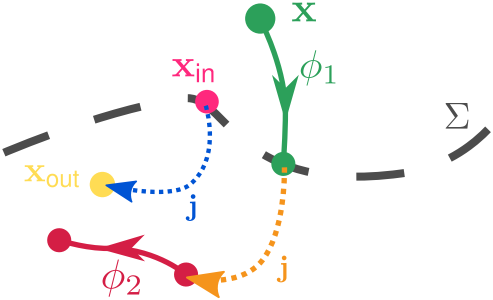

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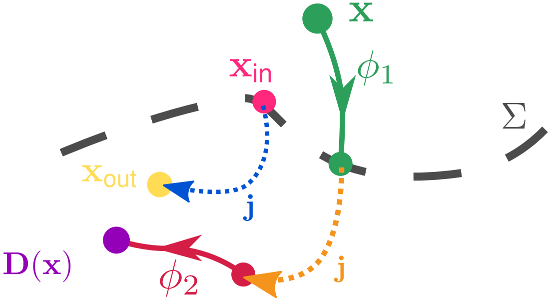

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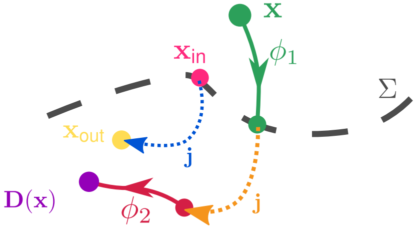

We can now write

$$\phi(\mathbf{x}_0, T) = \phi_2(\mathbf{D}(\phi_1(\mathbf{x}_0, t_{\text{ref}})), T - t_{\text{ref}}),$$

where the ZDM

$$\mathbf{D}(\mathbf{x}) = \phi_2(\mathbf{j}(\phi_1(\mathbf{x}, t(\mathbf{x}))), -t(\mathbf{x})).$$

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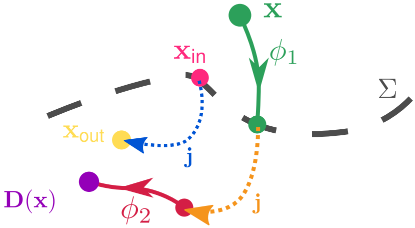

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Grazing Interactions

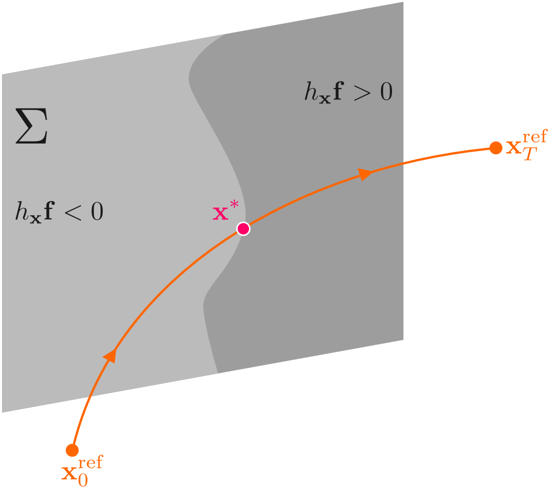

Figure: A grazing interaction in a hybrid system.

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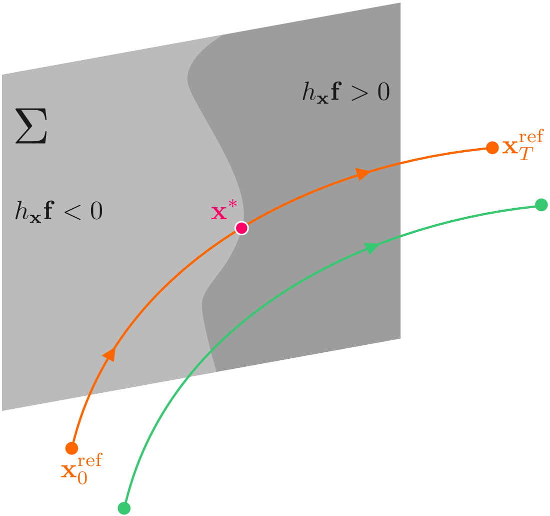

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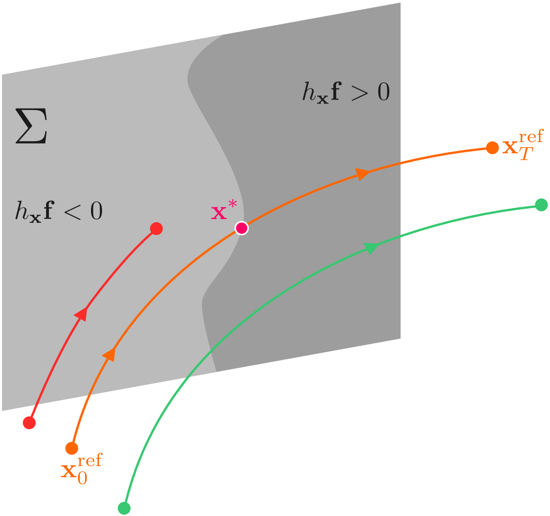

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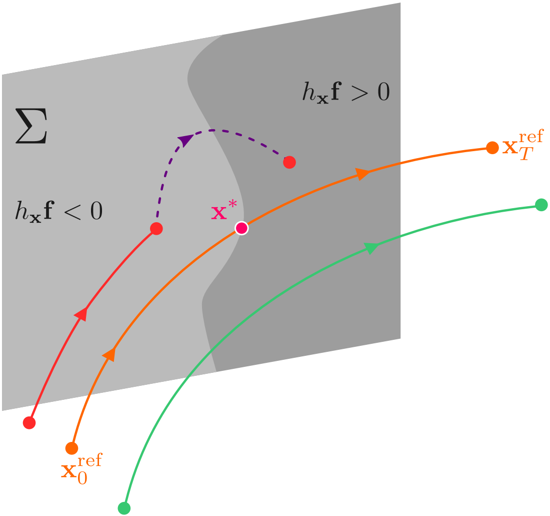

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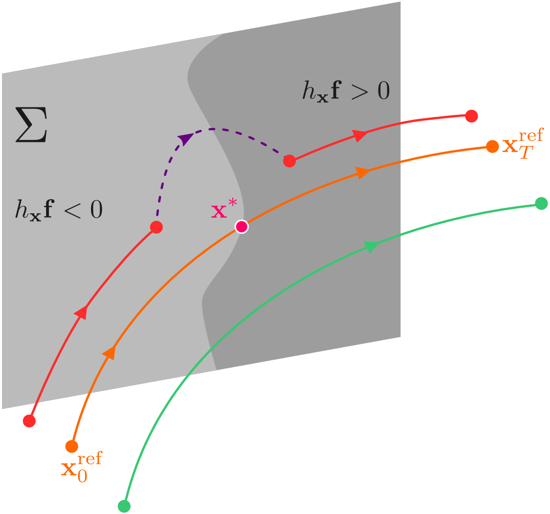

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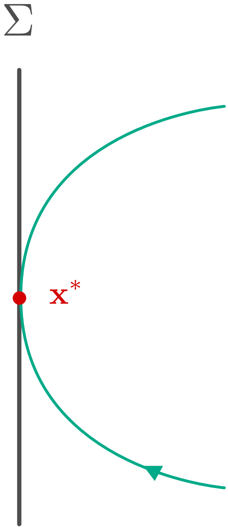

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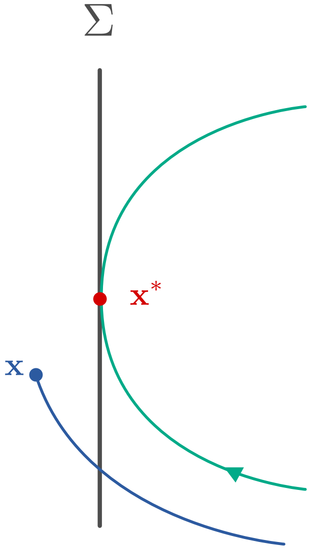

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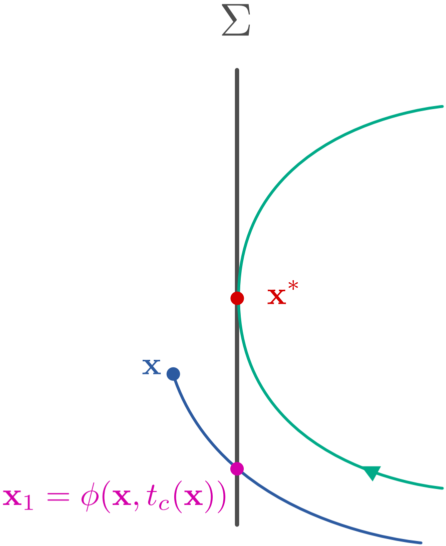

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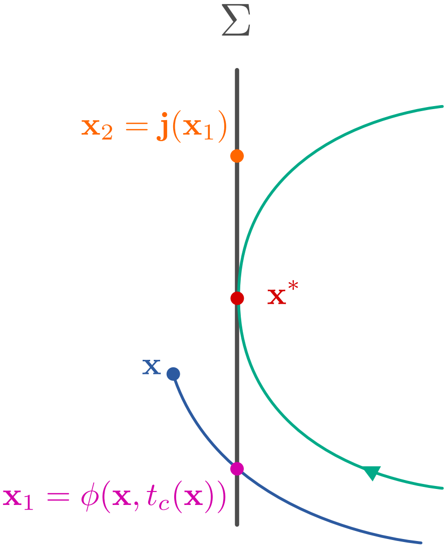

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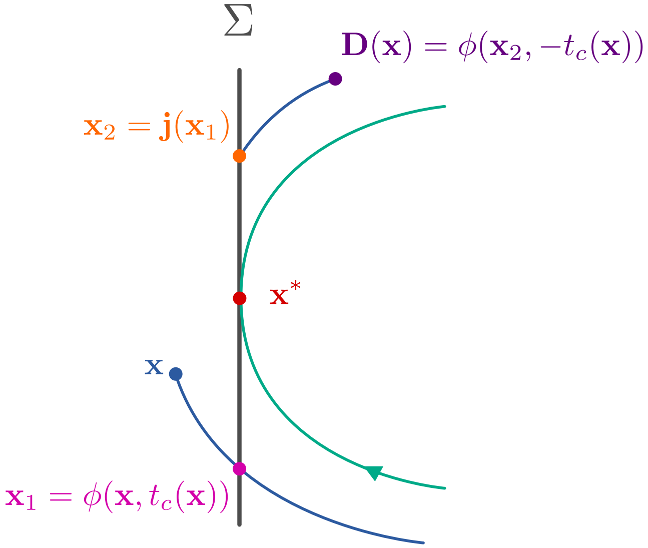

Figure: Constructing the grazing ZDM.

Types of Noise

Figure: A rugged boundary.

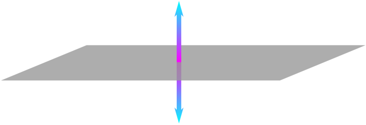

Figure: An oscillating boundary.

Rugged boundaries are suitable for modelling situations where the small-scale structure of the boundary is uncertain.

$$\tilde{\Sigma} = \{\mathbf{x} : \tilde{h}(\mathbf{x}, t) = 0\},$$
$$\tilde{h}(\mathbf{x}, t) = h(\mathbf{x}, t) - \chi(\mathbf{x}).$$

Oscillating boundaries are suitable for modelling situations where the boundary has small uncertain oscillations about a known mean.

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- have mean 0.

Transversal SZDM

For transversal crossings we base our approximation on linearisation about the corresponding trajectory in the deterministic system, taking

$$\Delta t_{\text{ref}} = \mathcal{P} / \left(h_{\mathbf{x}}(\mathbf{x}_{\text{in}}, t_{\text{ref}})\mathbf{f}_{\text{in}} + h_t(\mathbf{x}_{\text{in}}, t_{\text{ref}}) - \mathcal{V} \right),$$

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where $\mathcal{P} = \chi(\mathbf{x}_{\mathsf{in}})$, $\mathcal{V} = \chi_{\mathbf{x}}(\mathbf{x}_{\mathsf{in}})\mathbf{f}_{\mathsf{in}}$ in the rugged boundary case and $\mathcal{P} = P(t_{\mathsf{ref}})$, $\mathcal{V} = V(t_{\mathsf{ref}})$ in the case of an oscillating boundary.

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Example - Boucing a ball on a rugged oscillating floor.

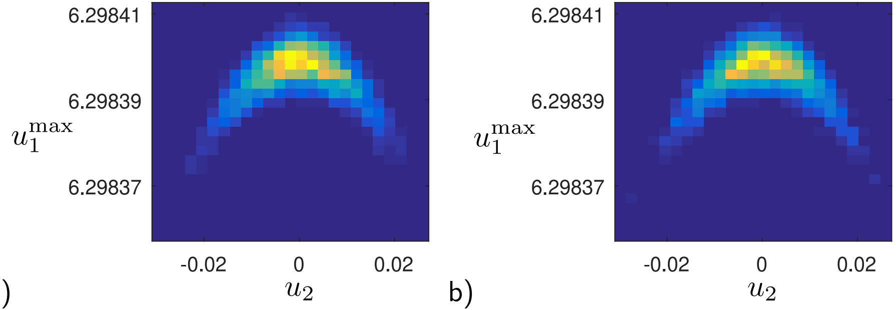

Figure: Heatmaps of the distribution of the maximum height attained by the bouncing ball and its corresponding horizontal position after one bounce on the rugged surface given by a) full simulation of the system b) linear approximation.

SZDMs for Higher-Order Discontinuities

When the vector field is $\mathcal{C}^{n-1}$ for $n \geq 1$ but has higher-order discontinuities the are are no linear effects so one must consider higher order approximations to capture the effects of noise and the crossing of a discontinuity boundary.

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$$\tilde{\mathbf{D}}(\mathbf{x}) \approx \mathbf{x} + \frac{g(\mathbf{x}^*)}{(n+1)h_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}}} \left(h(\mathbf{x})^{n+1} - \left(\frac{\mathcal{P}h_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}} - h(\mathbf{x})\mathcal{V}}{h_{\mathbf{x}}(\mathbf{x}_{\text{in}})\mathbf{f}_{\text{in}} - \mathcal{V}} \right)^{n+1} \right),$$

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where

$$g(\mathbf{x}) = \sum_{j=0}^{\infty} \frac{h^j}{(j+n)!} \frac{\partial^{j+n}}{\partial h^{j+n}} (\mathbf{f}_2 - \mathbf{f}_1)|_{h=0},$$

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Example - The Chua Circuit

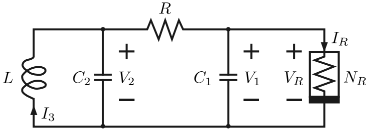

Figure: The Chua circuit.

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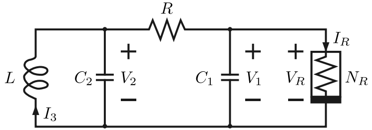

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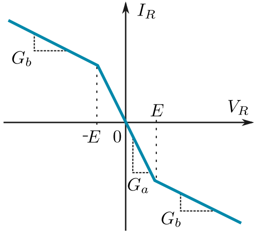

Figure: V-I characteristic.

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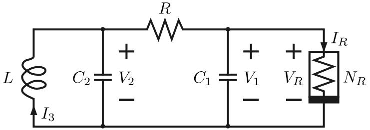

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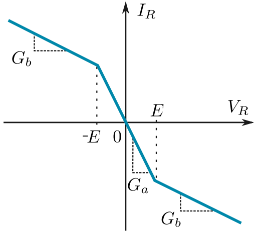

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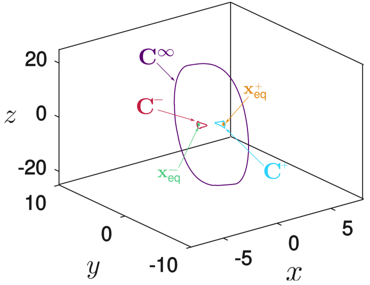

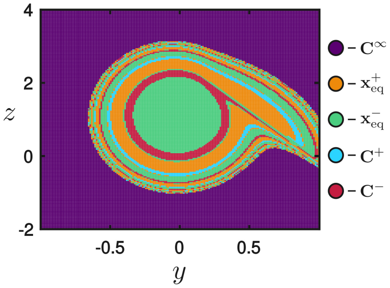

Figure: Coexisting attractors in the Chua circuit.

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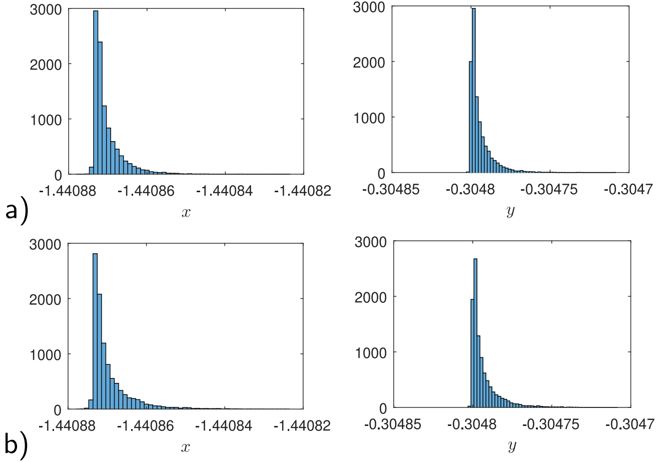

Figure: The Chua circuit with oscillating boundaries. The results of full numerical-simulation are shown in a) and the approximations obtained by using the SZDM in place of boundary interactions are shown in b).

Grazing SZDM

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Grazing SZDM

In the case of a grazing interaction we cannot linearise in the same way as we considered in the case of a transversal crossing. Instead we consider second-order approximations about the point and time where the deterministic component of $\tilde{h}$ (which we denote h) reaches its minimum value. We find that $\tilde{\mathbf{D}}(\mathbf{x}) = \mathbf{x} + \tilde{\Delta}\mathbf{x}$ where

$$\tilde{\Delta}\mathbf{x} \approx \sqrt{\left(-h_{\mathbf{x}}(\mathbf{x}^*)\mathbf{x} + \mathcal{P} + \mathcal{V}\frac{h_{\mathbf{x}}\mathbf{f}}{(h_{\mathbf{x}}\mathbf{f})_{\mathbf{x}}\mathbf{f}} + \frac{\mathcal{V}^2}{2(A_g - \mathcal{A})}\right) 2(A_g - \mathcal{A})\xi},$$

$\mathcal{P} = \chi$, $\mathcal{V} = \chi_{\mathbf{x}}\mathbf{f}$, $\mathcal{A} = (\chi_{\mathbf{x}}\mathbf{f})_{\mathbf{x}}\mathbf{f}$ in the rugged boundary case and $\mathcal{P} = P$, $\mathcal{V} = V$, $\mathcal{A} = A$ in the case of an oscillating boundary.

Example - A grazing impact oscillator

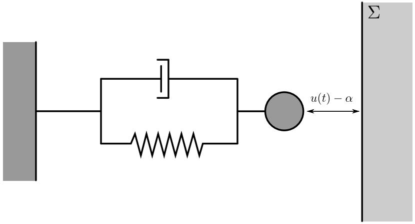

Figure: Schematic of a one-degree-of-freedom impact oscillator.

Example - A grazing impact oscillator

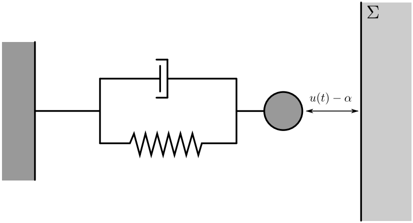

Figure: Schematic of a one-degree-of-freedom impact oscillator.

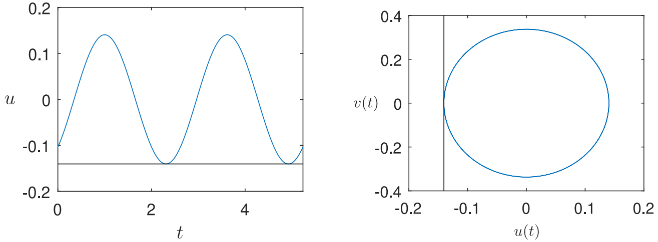

Figure: A sample grazing orbit.

Example - A grazing impact oscillator

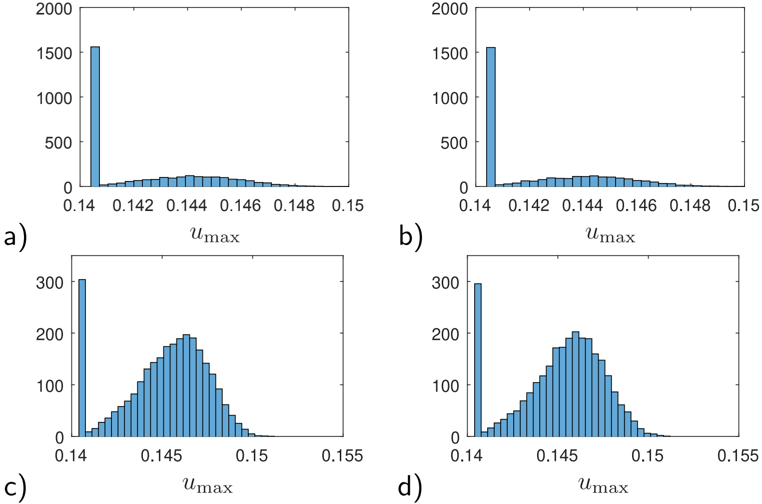

Figure: Histograms of the pdf of the maximum amplitude attained by the impact oscillator by a), c), full simulation of the system and b), d) approximation using the SZDM. a), b) $\varepsilon = 0$, c), d) $\varepsilon = 0.00005$.

Summary

Summary

- This work aims to further our knowledge of stochastic nonsmooth dynamical systems, an area which has so far seen limited research.
- Our work on the square root map highlights the complex and non-trivial effects noise can have on even the simplest nonsmooth systems.
  - ➥ Additive noise can both destroy and induce multistability.
- Our work on SZDMs led to the development and validation of new tools in analysing and efficiently simulating nonsmooth dynamical systems with noisy boundaries.
  - ➥ PWS systems.
  - ➥ Hybrid systems.
  - ➥ Systems with higher-order discontinuities.
  - ➥ Grazing in hybrid systems.

Thank you!

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