IRISH RESEARCH COUNCIL
An Chomhairle um Thaighde in Éirinn

NUI Galway
OÉ Gaillimh

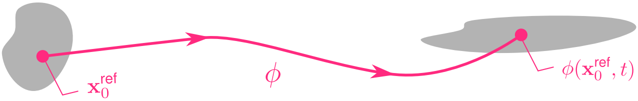
Boundary Noise in the Chua Circuit

Eoghan J. Staunton,
Petri T. Piiroinen

10th June 2019

eoghan.staunton@nuigalway.ie

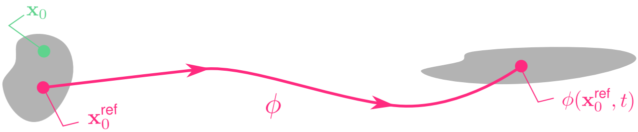
Linearisation



Suppose the system

$$\underline{x} = f(x); \quad x = x_0; \quad \text{has the unique solution} \quad x(t) = \quad (x_0; t): \qquad (1)$$

Linearisation

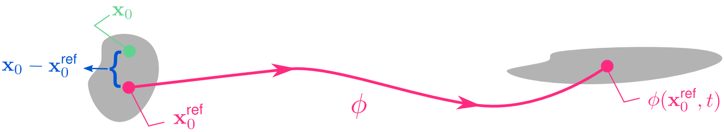


Suppose the system

$$\underline{x} = f(x); \quad x = x_0; \quad \text{has the unique solution} \quad x(t) = \; (x_0; t): \qquad (1)$$

Then for x_0 in a small neighbourhood of x_0^{ref}

Linearisation

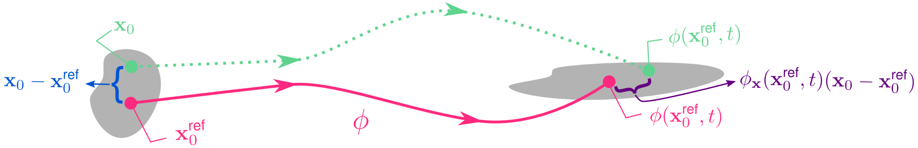


Suppose the system

$$\underline{x} = f(x); \quad x = x_0; \quad \text{has the unique solution} \quad x(t) = (x_0; t): \qquad (1)$$

Then for x_0 in a small neighbourhood of x_0^{ref}

Linearisation



Suppose the system

$$\underline{x} = f(x); \quad x = x_0; \quad \text{has the unique solution} \quad x(t) = \ (x_0; t). \qquad (1)$$

Then for x_0 in a small neighbourhood of x_0^{ref}

$$(x_0; t) \qquad (x_0^{\text{ref}}; t) = \ _{\mathbf{x}}(x_0^{\text{ref}}; t)(x_0 \quad x_0^{\text{ref}}) + O(kx_0 \quad x_0^{\text{ref}}k); \qquad (2)$$

where the Jacobian $_{\mathbf{x}}(x_0^{\text{ref}}; t)$ is the solution to the IVP

$$\_ = f_{\mathbf{x}}(\ (x_0^{\text{ref}}; t)) \quad ; \qquad (0) = \mathbf{I}. \qquad (3)$$

Linearisation

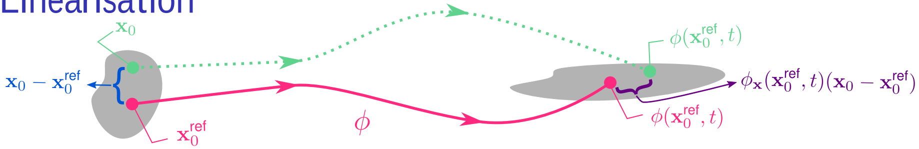


Figure: Linearisation of smooth dynamical systems

Linearisation

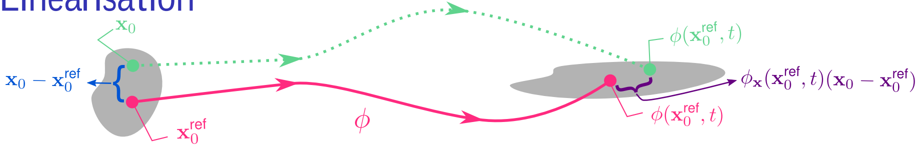


Figure: Linearisation of smooth dynamical systems

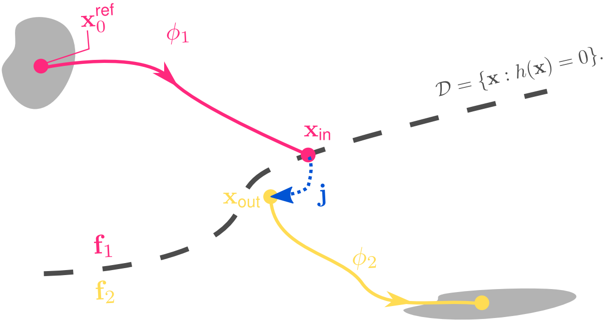


Figure: A nonsmooth dynamical system

Linearisation

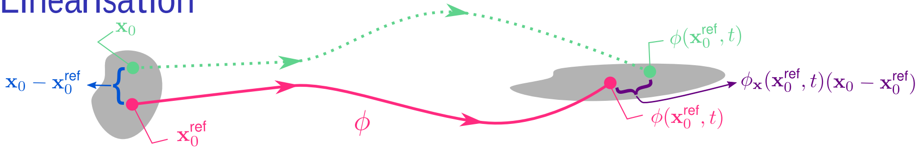


Figure: Linearisation of smooth dynamical systems

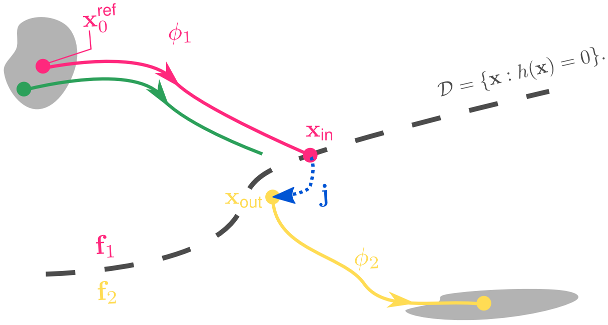


Figure: A nonsmooth dynamical system

Linearisation

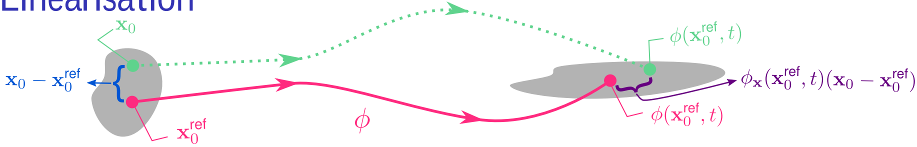


Figure: Linearisation of smooth dynamical systems

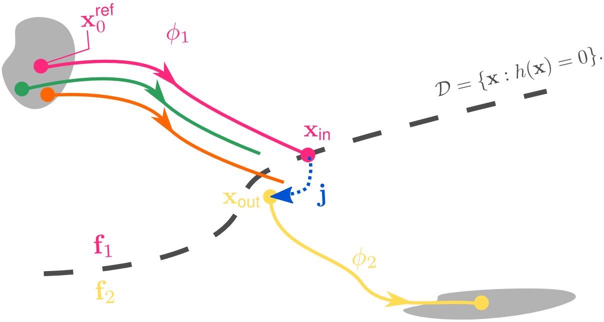


Figure: A nonsmooth dynamical system

Constructing the Zero-Time Discontinuity Mapping

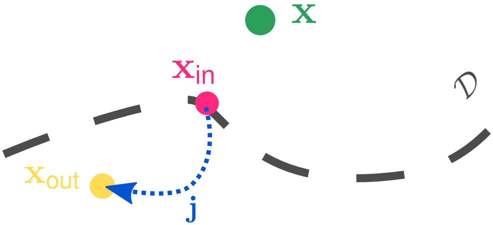


Figure: Constructing the ZDM

Constructing the Zero-Time Discontinuity Mapping

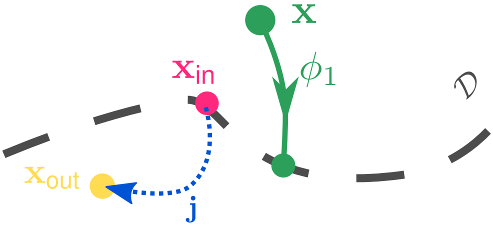


Figure: Constructing the ZDM

Constructing the Zero-Time Discontinuity Mapping

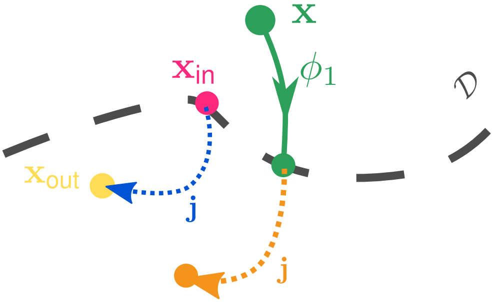


Figure: Constructing the ZDM

Constructing the Zero-Time Discontinuity Mapping

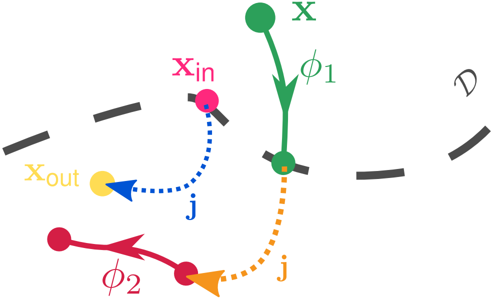


Figure: Constructing the ZDM

Constructing the Zero-Time Discontinuity Mapping

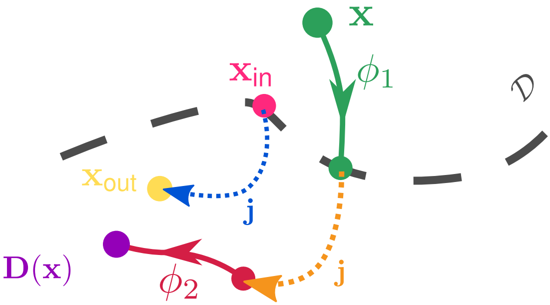


Figure: Constructing the ZDM

Constructing the Zero-Time Discontinuity Mapping

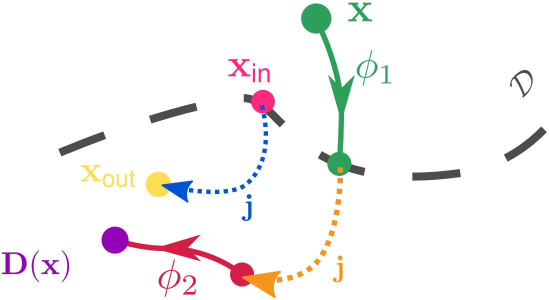


Figure: Constructing the ZDM

We can now write

$$(x_0; T) = {}_2(D({}_1(x_0; t_{ref})); T \quad t_{ref}); \qquad (4)$$

where the ZDM

$$D(x) = {}_2(j({}_1(x; t(x))); \quad t(x)) \qquad (5)$$

takes a point in a neighbourhood of x_{in} and maps it to a point in a neighbourhood of x_{out}.

The Saltation Matrix

The Jacobian of D evaluated at x_{in} is given by

$$\begin{aligned} D_{\mathbf{x}}(x_{in}) &= j_{\mathbf{x}}(x_{in}) + (j_{\mathbf{x}}(x_{in}) \quad f_{out}) t_{\mathbf{x}}(x_{in}) \\ &= j_{\mathbf{x}}(x_{in}) + \frac{(f_{out} \quad j_{\mathbf{x}}(x_{in}) f_{in}) h_{\mathbf{x}}(x_{in})}{h_{\mathbf{x}}(x_{in}) f_{in}}; \end{aligned} \tag{6}$$

where $f_{in} = f_1(x_{in})$ and $f_{out} = f_2(x_{out})$. In the case where h is explicitly time-dependent this becomes

$$D_{\mathbf{x}}(x_{in}) = j_{\mathbf{x}}(x_{in}) + \frac{(f_{out} \quad j_{\mathbf{x}}(x_{in}) f_{in}) h_{\mathbf{x}}(x_{in}; t_{ref})}{h_t(x_{in}; t_{ref}) + h_{\mathbf{x}}(x_{in}; t_{ref}) f_{in}}: \tag{7}$$

In both cases we have that

$$_{\mathbf{x}}(x_0^{ref}; T) = \ _{2;\mathbf{x}}(x_{out}; T \quad t_{ref}) D_{\mathbf{x}}(x_{in}) \ _{1;\mathbf{x}}(x_{in}; t_{ref}): \tag{8}$$

Introducing Noise

As in the deterministic case, we define the discountinuity boundary D as the zeros of a function h. For a stochastically oscillating boundary we let h take the form

$$h(x;t) = \hat{h}(x;t) \quad P(t); \tag{9}$$

where the function $\hat{h}$ is deterministic and $P(t)$ is a stochastic process.

Introducing Noise

As in the deterministic case, we define the discountinuity boundary D as the zeros of a function h. For a stochastically oscillating boundary we let h take the form

$$h(\mathbf{x};t) = \hat{h}(\mathbf{x};t) \quad P(t); \tag{9}$$

where the function $\hat{h}$ is deterministic and $P(t)$ is a stochastic process. We further require that P is a mean reverting stochastic process that has mean 0, is at least once differentiable and does not depend on $\mathbf{x}$.

Introducing Noise

As in the deterministic case, we define the discountinuity boundary D as the zeros of a function h. For a stochastically oscillating boundary we let h take the form

$$h(\mathrm{x}; t) = \hat{h}(\mathrm{x}; t) \quad P(t); \tag{9}$$

where the function $\hat{h}$ is deterministic and $P(t)$ is a stochastic process. We further require that P is a mean reverting stochastic process that has mean 0, is at least once differentiable and does not depend on x.

Let $\hat{t}_{\mathrm{ref}}$ be the time of flight from $\mathrm{x}_0^{\mathrm{ref}}$ to the boundary in the absence of noise, i.e.

$$\hat{h}(\ _1(\mathrm{x}_0^{\mathrm{ref}}; \hat{t}_{\mathrm{ref}})) = 0. \tag{10}$$

Introducing Noise

As in the deterministic case, we de ne the discountinuity boundary D as the zeros of a function h. For a stochastically oscillating boundary we let h take the form

$$h(\mathrm{x}; t) = \hat{h}(\mathrm{x}; t) \quad P(t); \tag{9}$$

where the function $\hat{h}$ is deterministic and $P(t)$ is a stochastic process. We further require that P is a mean reverting stochastic process that has mean 0, is at least once di erentiable and does not depend on x.

Let $\hat{t}_{\mathrm{ref}}$ be the time of ight from $\mathrm{x}_0^{\mathrm{ref}}$ to the boundary in the absence of noise, i.e.

$$\hat{h}(\;_1(\mathrm{x}_0^{\mathrm{ref}}; \hat{t}_{\mathrm{ref}})) = 0: \tag{10}$$

We de ne $\quad t_{\mathrm{ref}}$ to be the random variable given by the di erence between $\hat{t}_{\mathrm{ref}}$ and the actual time of ight

$$t_{\mathrm{ref}} = t_{\mathrm{ref}} \quad \hat{t}_{\mathrm{ref}}: \tag{11}$$

Stochastic Saltation

In order to deal with stochastically oscillating boundaries we extend the state space, such that the state vector and vector field are given by

$$\tilde{\mathbf{x}} = (\mathbf{x}, t, t_{\mathrm{ref}})^T \quad \text{and} \quad \tilde{\mathbf{f}} = (\mathbf{f}, 1, 0)^T, \tag{12}$$

respectively.

Stochastic Saltation

In order to deal with stochastically oscillating boundaries we extend the state space, such that the state vector and vector eld are given by

$$\tilde{\mathrm{x}} = (\mathrm{x}; t; \quad t_{\mathrm{ref}})^T \quad \text{and} \quad \tilde{\mathrm{f}} = (\mathrm{f}; 1; 0)^T; \tag{12}$$

respectively.

We calculate the saltation matrix in this extended state space before projecting back. As a result, in the original state space we nd that

$$(\mathrm{x}_0; t) \quad (\hat{\mathrm{x}}_0^{\mathrm{ref}}; t) \quad {}_{\mathbf{x}}(\hat{\mathrm{x}}_0^{\mathrm{ref}}; t)(\mathrm{x}_0 \quad \hat{\mathrm{x}}_0^{\mathrm{ref}}) + {}_{2;\mathbf{x}}(\hat{\mathrm{x}}_{\mathrm{out}}; t \quad \hat{t}_{\mathrm{ref}})(\hat{\mathrm{f}}_{\mathrm{in}} \quad \hat{\mathrm{f}}_{\mathrm{out}}) \quad t_{\mathrm{ref}}; \tag{13}$$

where

$${}_{\mathbf{x}}(\hat{\mathrm{x}}_0^{\mathrm{ref}}; t) = {}_{2;\mathbf{x}}(\hat{\mathrm{x}}_{\mathrm{out}}; t \quad \hat{t}_{\mathrm{ref}}) \mathrm{D}_{\mathbf{x}}(\hat{\mathrm{x}}_{\mathrm{in}}) \; {}_{1;\mathbf{x}}(\hat{\mathrm{x}}_{\mathrm{in}}; \hat{t}_{\mathrm{ref}}) \tag{14}$$

and

$$\mathrm{D}_{\mathbf{x}}(\hat{\mathrm{x}}_{\mathrm{in}}) = \mathrm{I} + \frac{(\hat{\mathrm{f}}_{\mathrm{out}} \quad \hat{\mathrm{f}}_{\mathrm{in}})\hat{h}_{\mathbf{x}}(\hat{\mathrm{x}}_{\mathrm{in}}; \hat{t}_{\mathrm{ref}})}{\hat{h}_{\mathbf{x}}(\hat{\mathrm{x}}_{\mathrm{in}}; \hat{t}_{\mathrm{ref}})\hat{\mathrm{f}}_{\mathrm{in}} + \hat{h}_t(\hat{\mathrm{x}}_{\mathrm{in}}; \hat{t}_{\mathrm{ref}}) \quad V(\hat{t}_{\mathrm{ref}} | P(\hat{t}_{\mathrm{ref}}) = 0)}: \tag{15}$$

In all the above ^ indicates the values associated with the deterministic reference trajectory.

Summary

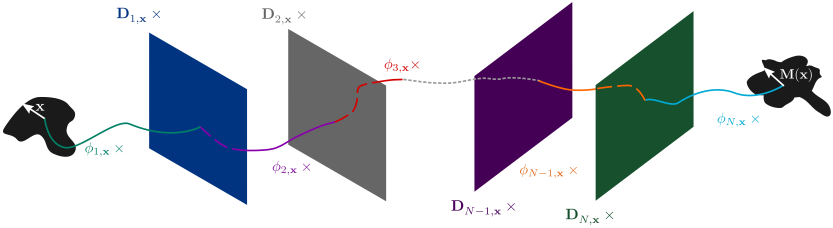


Figure: Linearising Discontinuous Systems

Summary

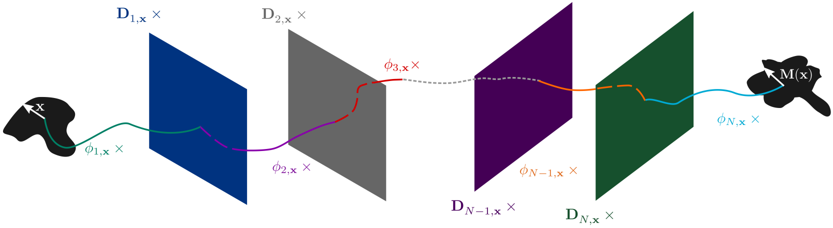
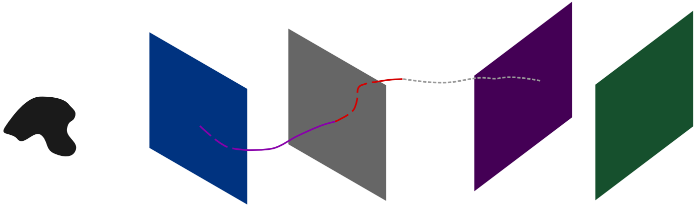


Figure: Linearising Discontinuous Systems



The Chua Circuit

Figure: The Chua Circuit

The Chua Circuit

Figure: The Chua Circuit

Figure: TheV I characteristic of the Chua Diode.

The Chua Circuit

Figure: The Chua Circuit

Created with the aim of being the simplest autonomous circuit capable of generating chaos [Mat84, Chu92].

Figure: TheV I characteristic of the Chua Diode.

The Chua Circuit

Figure: The Chua Circuit

- Created with the aim of being the simplest autonomous circuit capable of generating chaos [Mat84, Chu92].
- First physical system for which the presenc of chaos was shown experimentally, numerically and mathematically [CKM86].

Figure: The V I characteristic of the Chua Diode.

The Chua Circuit

Figure: The Chua Circuit

- Created with the aim of being the simplest autonomous circuit capable of generating chaos [Mat84, Chu92].
- First physical system for which the presenc of chaos was shown experimentally, numerically and mathematically [CKM86].
- Contains four linear elements and one nonlinear resistor known as Chua diode.

Figure: The V I characteristic of the Chua Diode.

The Chua Circuit

Figure: The Chua Circuit

Figure: TheV I characteristic of the Chua Diode.

- Created with the aim of being the simplest autonomous circuit capable of generating chaos [Mat84, Chu92].
- First physical system for which the presenc of chaos was shown experimentally, numerically and mathematically [CKM86].
- Contains four linear elements and one nonlinear resistor known as a Chua diode.
- Easily and cheaply constructed using stadard electronic components [Ken92].

System equations

The dynamics of the Chua circuit can be described by the following nondimensionalised state equations

$$\begin{aligned} \frac{dx}{dt} &= \quad (y \quad x \quad g(x))\,; \\ \frac{dy}{dt} &= x \quad y + z; \\ \frac{dz}{dt} &= \quad (\ y \ + \ z\,); \end{aligned} \tag{16}$$

where $g(x)$ is the piecewise linear function representing the V-I characteristic of Chua's diode

$$g(x) = \begin{cases} m_1 x + m_1 \quad m_0 & \text{if } x < \quad 1; \\ (m_0 \quad)x & \text{if } |x| \quad 1; \\ m_1 x + m_0 \quad m_1 & \text{if } x > 1. \end{cases} \tag{17}$$

Complicated Dynamics

Figure: A Zoo of Attractors Produced by the Chua Circuit [BP08]

Hidden and Self-Excited Attractors

Hidden attractors: have basins of attraction that do not intersect with small neighborhoods of equilibria.

Self-excited attractors : Can be found by following trajectories from the neighbourhoods of unstable equilibria until the end of a transien process [LK13].

Hidden and Self-Excited Attractors

- Hidden attractors: have basins of attraction that do not intersect with small neighborhoods of equilibria.
- Self-excited attractors : Can be found by following trajectories from the neighbourhoods of unstable equilibria until the end of a transien process [LK13].

For a range of parameter values the Chua circuit has a 5-stable regime including 3 hidden periodic attractors.

A Discontinuous Model

Provided the magnitude of is not too large the hidden attractors in the 5-stable regime continue to exist and can be easily found by numerical continuation.

They are destroyed in saddle-bifurcations if the magnitude of grows too large.

Figure: Bifurcation diagram showing the saddle bifurcations of C as the magnitude of grows. Here = 8:4, = 12, = 0:005, m_0 = 1:2 and m_1 = 0:145.

Steady-State Distributions

Figure: Steady state distribution of orbit errors on the discontinuity boundary D for trajectories with initial condition on the periodic orbit C .

Figure: Convergence of $_z$ to its steady state value for the distribution shown on the left.

Destroying Periodic Attractors

Destroying Periodic Attractors

A 3-Stable Regime

For a range of parameter values the Chua circuit has a 3-stable regime including 2 symmetric periodic attractors.

A 3-Stable Regime

For a range of parameter values the Chua circuit has a 3-stable regime including 2 symmetric periodic attractors.

The two periodic attractors merge in a supercritical pitchfork bifurcation if the magnitude grows too large.

Merging Periodic Attractors

Merging Periodic Attractors

Conclusions

The concept of a saltation matrix can be generalised to stochastic systems.

Conclusions

- The concept of a saltation matrix can be generalised to stochastic systems.
- This can be used to estimate the dynamics of discontinuous system with noisy boundaries.
  - destruction of attractors
  - multi/monostability
  - merging of attractors
  - ickering/switching

Conclusions

- The concept of a saltation matrix can be generalised to stochastic systems.
- This can be used to estimate the dynamics of discontinuous systems with noisy boundaries.
  - destruction of attractors
  - multi/monostability
  - merging of attractors
  - ickering/switching
- It remains to generalise our method
  - Second order terms for continuous systems
  - Non-identity boundary mappings
  - Dealing with non-transversal intersections

- Eleonora Bilotta and Pietro Pantano, *A gallery of chua attractors*, vol. 61, World Scienti c, 2008.
- Leon O Chua, *The genesis of chua's circuit*, International Journal of Electronis Communication **46** (1992), no. 4, 250{257.
- Leon O Chua, Motomasa Komuro, and Takashi Matsumoto, *The double scroll family*, IEEE transactions on circuits and systems **33** (1986), no. 11, 1072{1118.
- Mario Di Bernardo, Chris J Budd, Alan R Champneys, Piotr Kowalczyk, Arne B Nordmark, Gerard Olivar Tost, and Petri T Piiroinen, *Bifurcations in nonsmooth dynamical systems*, SIAM review **50** (2008), no. 4, 629{701.
- Michael Peter Kennedy, *Robust op amp realization of chua's circuit*, Frequenz **46** (1992), no. 3-4, 66{80.

- Gennady A Leonov and Nikolay V Kuznetsov, *Hidden attractors in dynamical systems. from hidden oscillations in hilbert{kolmogorov, aizerman, and kalman problems to hidden chaotic attractor in chua circuits*, International Journal of Bifurcation and Chaos **23** (2013), no. 01, 1330002.
- Takashi Matsumoto, *A chaotic attractor from chua's circuit*, IEEE Transactions on Circuits and Systems **31** (1984), no. 12, 1055{1058.
- Eoghan J Staunton and Petri T Piiroinen, *Discontinuity mappings for stochastic nonsmooth systems*, In Preparation (2019).
- ______, *Estimating the dynamics of systems with noisy boundaries*, Submitted (2019).