Dynamics of multistable networks

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Multistable systems

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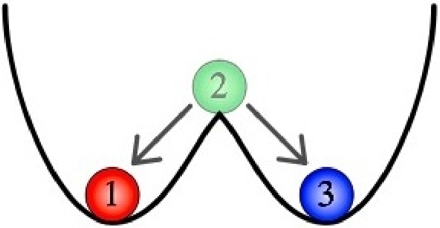

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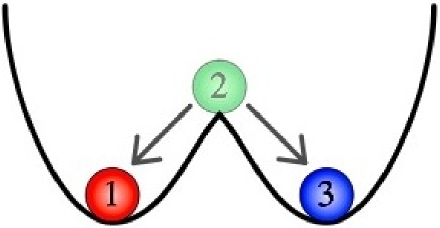

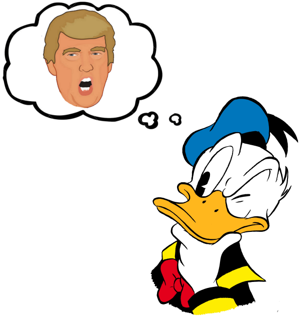

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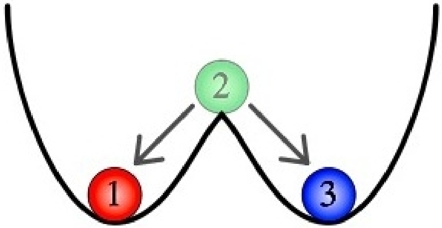

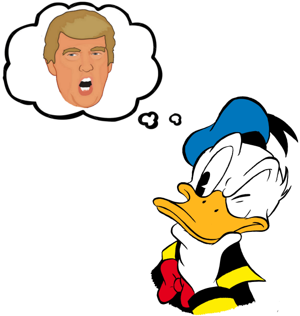

Similarly, a dynamical system with more than two stable equilibrium points is **multistable**.

Normal form of a supercritical pitchfork bifurcations

The simplest bistable system is:

$$\dot{x} = -x^3 + bx \tag{1}$$

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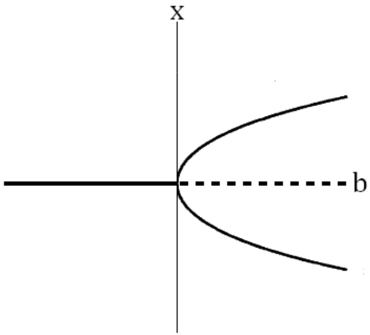


$$0 \text{ if } b \leq 0 \qquad \{0, +\sqrt{b}, -\sqrt{b}\} \text{ if } b > 0 \tag{2}$$

Key questions

The research focuses on the two following problems:

Bifurcation diagram of a network of supercritical pitchforks

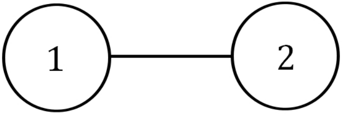
What happens when several pitchforks (1) are connected to each other? How many stable equilibrium points are there? How does the type and the strength of the coupling affect the stability?

Control of multistable network

Is it possible to trigger the transition from a stable equilibrium point to another? How many and which nodes should be controlled in order to drive the whole network from a stable state to another?

Two coupled supercritical pitchforks

A common and easy way to connect two nodes is using a **linear coupling**.



$$\begin{aligned} \dot{x}_1 &= -x_1^3 + bx_1 + \sigma(x_2 - x_1) \\ \dot{x}_2 &= -x_2^3 + bx_2 + \sigma(x_1 - x_2) \end{aligned} \tag{3}$$

Where σ is the strength of the coupling.

Global equilibrium points

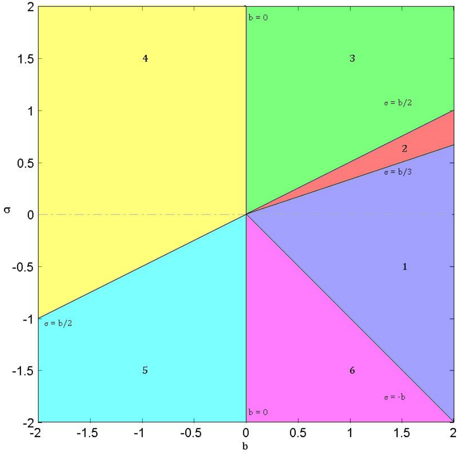
System (3) can have up to 9 equilibrium points. A closed solution for all of them is available:

$$
\begin{aligned}
\mathbf{X}_0 &= (0,0) \\
\mathbf{X}_1 &= \left(\sqrt{b}, \sqrt{b}\right) \\
\mathbf{X}_2 &= \left(-\sqrt{b}, -\sqrt{b}\right) \\
\mathbf{X}_3 &= \left(\sqrt{b-2\sigma}, -\sqrt{b-2\sigma}\right) \\
\mathbf{X}_4 &= \left(-\sqrt{b-2\sigma}, \sqrt{b-2\sigma}\right) \\
\mathbf{X}_5 &= \left(f_{x_1}^{(5)}(b,\sigma), f_{x_2}^{(5)}(b,\sigma)\right) \\
\mathbf{X}_6 &= \left(f_{x_1}^{(6)}(b,\sigma), f_{x_2}^{(6)}(b,\sigma)\right) \\
\mathbf{X}_7 &= \left(f_{x_1}^{(7)}(b,\sigma), f_{x_2}^{(7)}(b,\sigma)\right) \\
\mathbf{X}_8 &= \left(f_{x_1}^{(8)}(b,\sigma), f_{x_2}^{(8)}(b,\sigma)\right)
\end{aligned} \tag{4}
$$

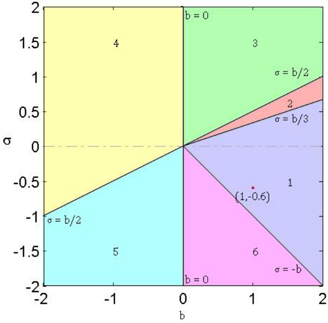
Although the existence and the stability of these equilibrium points is strictly dependant on the values of b and σ.

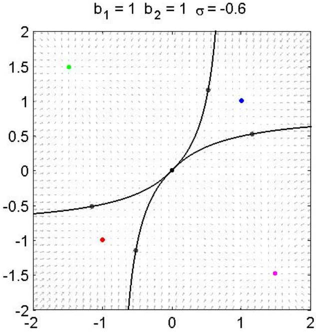
Bifurcations diagrams

Analytical and numerical analysis led to a bifurcation diagram (b, σ) with 6 regions.

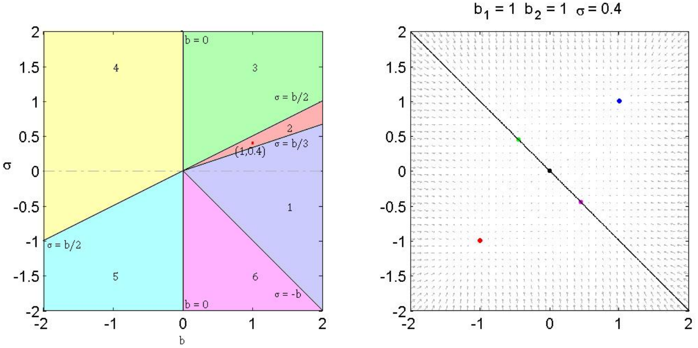


Region 1: quadstability

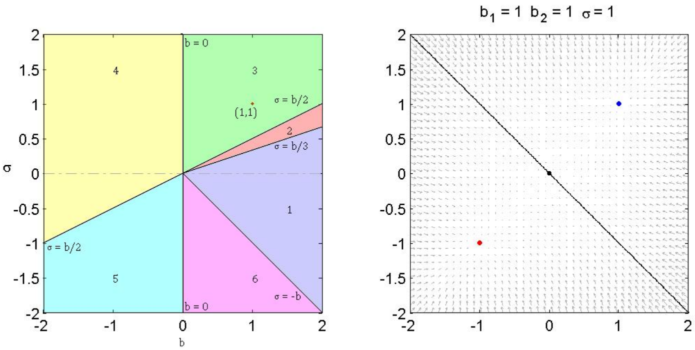




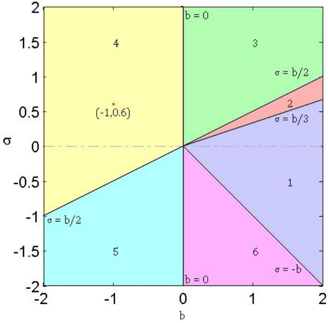
Region 2 : bistability

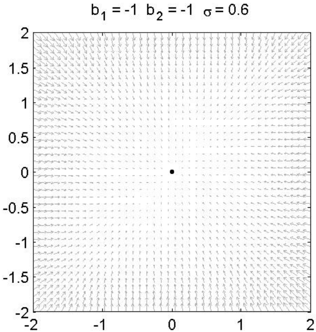


Region 3 : bistability

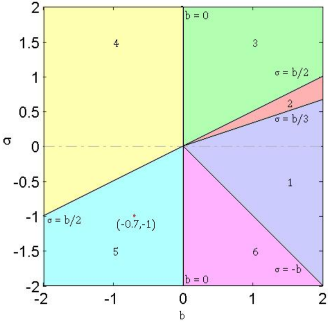


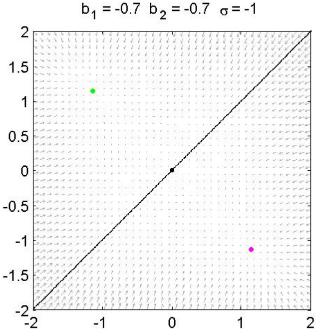
Region 4 : monostability



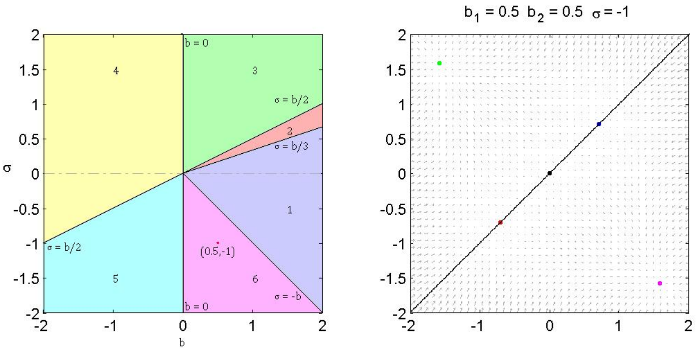


Region 5 : bistability





Region 6 : bistability



Conclusions

- The bifurcation diagram gives a complete scenario of when multistability is achievable for any values of b and σ.
- The bounds between the region are **bifurcation points**, i.e. points where the qualitative behaviour of the system changes.

Forthcoming research

- **1:** Structural stability for pitchfork networks with N nodes.

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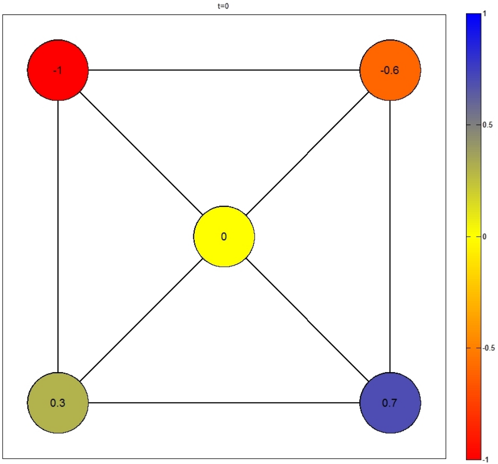

Play video *5NDynamics.avi*

Forthcoming research

- **2:** Control of pitchfork netoworks with N nodes.

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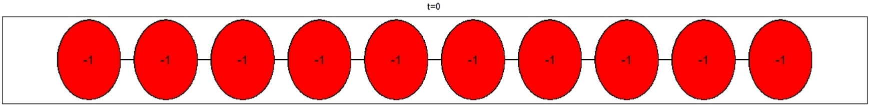

Play video *10NControl.avi*

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