Convex optimization: an introduction

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Overview

Mathematical optimization

Definition (Mathematical optimization problem)

A mathematical optimization problem has the form:

$$\begin{aligned} &\text{minimize } f_0(\mathbf{x}) \\ &\text{subject to } f_i(\mathbf{x}) \leq b_i \quad i = 1, \cdots, n \end{aligned} \tag{1}$$

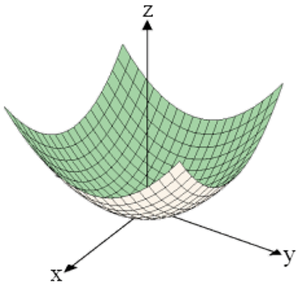
$\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is the **optimization variable**.

$f_0 : \mathbb{R}^n \to \mathbb{R}$ is the **objective function**.

$f_i : \mathbb{R}^n \to \mathbb{R} \quad i = 1, \cdots, n$ are the **inequality constraint functions**.

$b_1, \cdots, b_m \in \mathbb{R}$ are the **limits** or **bounds** for the contraints.

Optimal vector



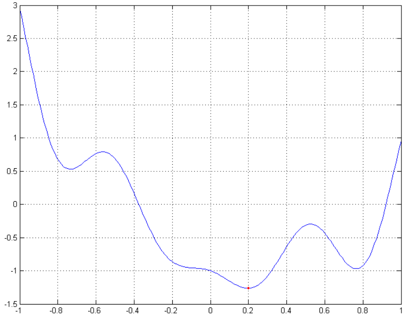
Definition (Optimal vector)

A vector $\mathbf{x}^*$ is called **optimal** (or solution of the optimization problem) if

$$\forall \mathbf{z} : f_1(\mathbf{z}) < b_1, \cdots, f_m(\mathbf{z}) < b_m \Rightarrow f_0(\mathbf{z}) \geq f_0(\mathbf{x}^*) \tag{2}$$

Optimization in $\mathbb{R}$

Let us consider a class C^2 function $f(x)$ with $x \in \mathbb{R}$



Minima are the points x^* such that:

$$\frac{d}{dx} f(x)|_{x=x^*} = 0 \quad \wedge \quad \frac{d^2}{dx^2}|_{x=x^*} \geq 0 \tag{3}$$

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- The optimal might not be found in reasonable time.

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Least-squares problems

A least square problem is an optimization problem with no contraints, in which the objective is a quadratic function.

Definition (Least-squares problem)

A least-squares problem has the form:

$$\text{minimize} f_0(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 = \sum_{i=1}^{k} (\mathbf{a}_i^T \mathbf{x} - b_i)^2 \tag{4}$$

where $\mathbf{A} \in \mathbb{R}^{k \times n}$ with $k \geq n$.

The **solution** can be reduced to solving a set of linear equations:

$$(\mathbf{A}^T\mathbf{A})\mathbf{x} = \mathbf{A}^T\mathbf{b} \Rightarrow \mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \tag{5}$$

The computation time to solve a least-squares problem is about $n^2 k$.

Linear programming

Linear programming is an important class of optimization problems in which the objective and all the constraints are linear.

Definition (Linear programming)

A linear programming problem has the form:

$$\begin{aligned} &\text{minimize } f_0(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \\ &\text{subject to } f_i(\mathbf{x}) = \mathbf{a}^T\mathbf{x}_i \leq b_i \quad i = 1, \cdots, n \end{aligned} \tag{6}$$

where $\mathbf{A} \in \mathbb{R}^{k \times n}$ with $k \geq n$.

There are several efficient methods to solve a linear programming problem.
The most common one is the **simplex**.
The computation time to solve a least-squares problem is about n^2m if $m \geq n$.

Convex optimization problems

Convex optimization problems is a more general family of optimization problem. It includes least-squares and linear-programming.

Definition (Convex optimization problem)

A convex optimization problem has the form:

$$\begin{aligned} &\text{minimize } f_0(\mathbf{x}) \\ &\text{subject to } f_i(\mathbf{x}) \leq b_i \quad i = 1, \cdots, n \end{aligned} \tag{7}$$

where the functions $f_0(\mathbf{x}), f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})$ are all **convex**, i.e.

$$\begin{gathered} f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y}) \\ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \wedge \forall \alpha, \beta \in \mathbb{R}, \text{ such that } \alpha \geq 0, \beta \geq 0 \wedge \alpha + \beta = 1 \end{gathered} \tag{8}$$

There is not a general solution for convex optimization problems. However there are some effective methods.

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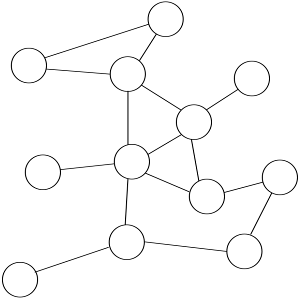
- Join Complex Networks and Convex Optimization theories.

Networks and convex optimization

Let us consider a generic Network. Assume that we have a given fixed amount of resources to allocate among the nodes.



The aim is to formulate a proper convex optimization problem.

References

Stephen Boyd and Lieven Vandenberghe (2004)
Convex Optimization
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