Analysis of multistable networks

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2. Multistable networks
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   - Analysis of complex networks
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Multistability: Definition

Consider a dynamical system

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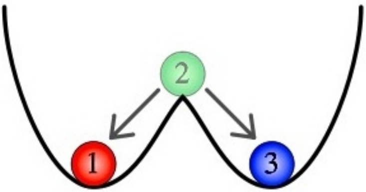
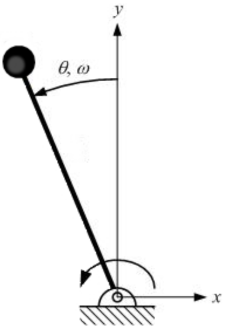
We say that the system is K-**stable** if it has exactly K attractors.

Multistability: Examples

2-stable (or bistable) systems are very common:

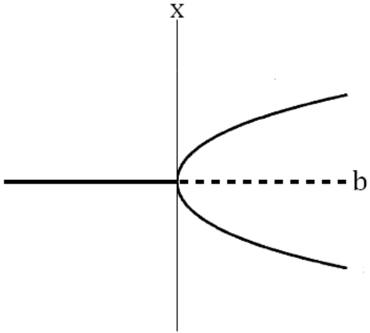




Multistability: Supercritical pitchfork bifurcation

The simplest bistable system is:

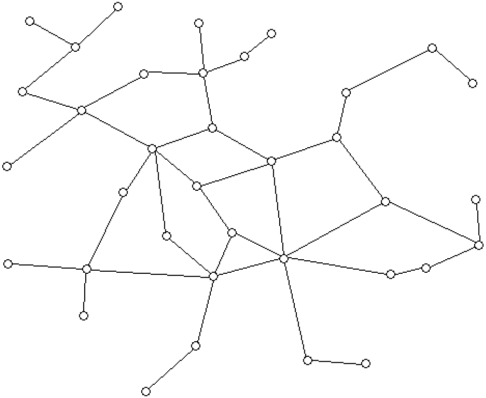
$$\dot{x} = -x^3 + bx \tag{2}$$



$$0 \text{ if } b \leq 0 \qquad \{0, +\sqrt{b}, -\sqrt{b}\} \text{ if } b > 0 \tag{3}$$

Multistable networks: Coupled cubic systems

Consider a Network with N nodes,



each of which has an internal dynamics described by (2), connected with a linear coupling σ according to a given structure,

Multistable networks: Coupled cubic systems

The dynamics of each node is given by

$$\dot{x}_i = -x_i^3 + bx_i + \sigma \sum_{j \neq i}^{N} a_{ij}(x_j - x_i) \tag{4}$$

where

$$a_{ij} = \begin{cases} 1 & \text{if there is and edge between the nodes } i \text{ and } j \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

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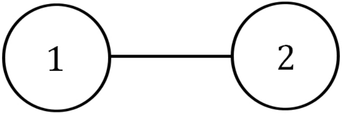
Key question

How many global equilibrium points does the network have for any value of b and σ?

The simplest case is with only two nodes:



whose dynamics is given by

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + bx_1 + \sigma(x_2 - x_1)\\ \dot{x}_2 &= -x_2^3 + bx_2 + \sigma(x_1 - x_2)\end{aligned} \tag{6}$$

Multistable networks: Analysis of a simple network

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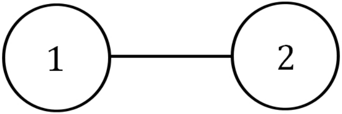

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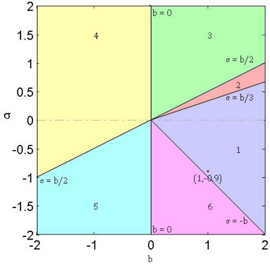
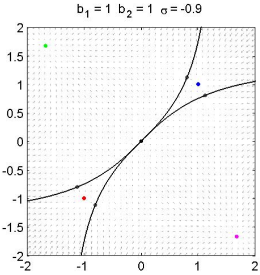
According to the value of b and σ we can have 1,3,5, or 9 equilibrium points, for all of which we have a closed solution.

Multistable networks: Analysis of a simple network





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We can not proceed either analytically or numerically. What do we do?

Take a network and get rid of all the edges ($\sigma = 0$),

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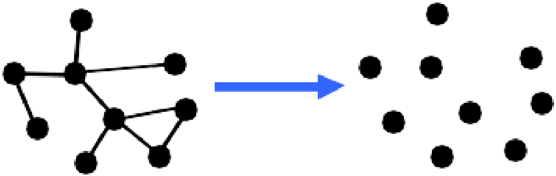

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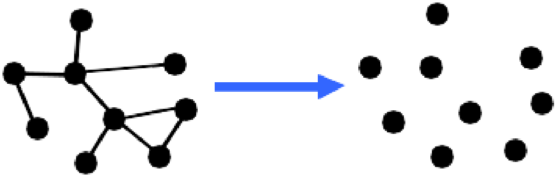

Assuming $b > 0$ for each node there are exactly three equibrium points:

$$\begin{array}{ll} \sqrt{b} & \text{stable} \\ -\sqrt{b} & \text{stable} \\ 0 & \text{unstable} \end{array} \quad (7)$$

Global equilibria are therefore

$$\{-\sqrt{b}, 0, \sqrt{b}\} \times \{-\sqrt{b}, 0, \sqrt{b}\} \times \ldots \times \{-\sqrt{b}, 0, \sqrt{b}\} \tag{8}$$

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I.E. 3^N equilibrium points that can be classified thanks to the analogy with an hypercube

$$\begin{array}{lll} 2^N & \text{stable nodes} & (\pm 1, \pm 1, \ldots, \pm 1) \\ 2^{N-m}\binom{N}{m} & \text{saddles with } m \text{ unstable manifolds} & x_i = 0 \text{ for } m \text{ components} \\ 1 & \text{unstable node} & (0, 0, \ldots, 0) \end{array} \tag{9}$$

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Bifurcation points for each of these equilibria can be found using a **continuation method**, that allows to follow branches of solution in the augmented space $\{\mathbf{x}, \sigma\} \in \mathbb{R}^{N+1}$.

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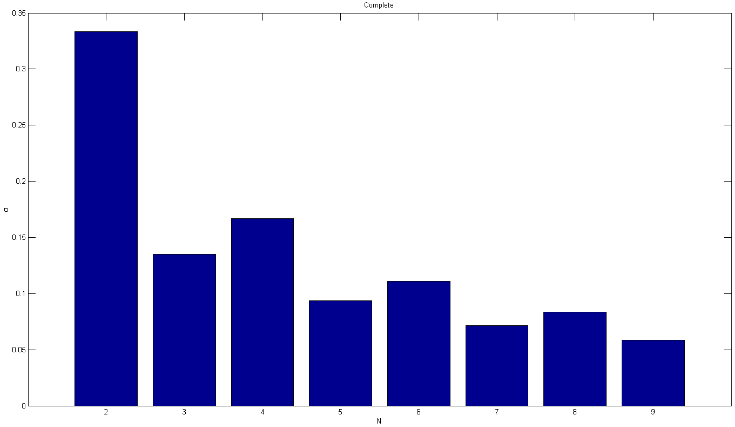
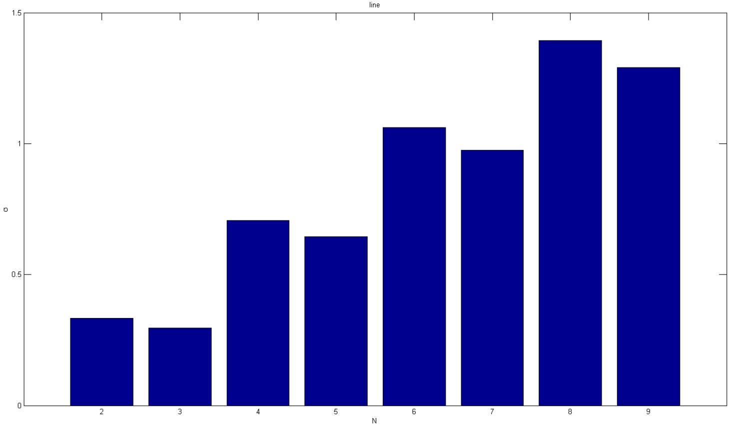
Matlab toolbox **Computational Continuation Core** (or CoCo) has been used to determine the minimum value of σ that guarantees 2-stability, for networks with 2 to 9 nodes and a given topology.

Complete network



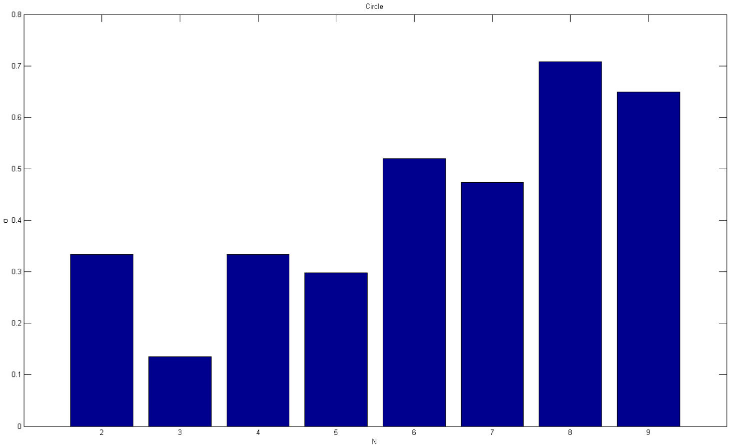
Multistable networks: Results

Line network



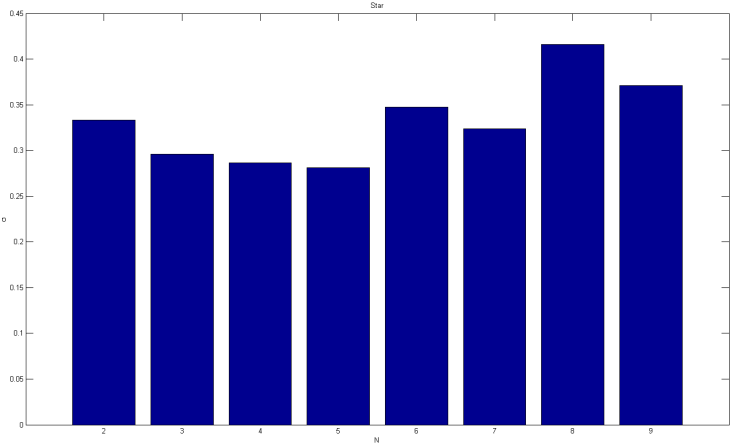
Multistable networks: Results

Circle network



Multistable networks: Results

Star network



Forthcoming research

- Classification of topologies
- Generalization of the results to any structure
- Detection of sub-regions of interest
- Control of the network

References

Harry Dankowicz and Frank Schilder (2013)
Recipes for Continuation
ISBN 978-1-61-1972-56-6
SIAM

Frank C. Hoppensteadt and Eugene M. Izhikevich (1997)
Weakly Connected Neural Networks
ISBN 0-387-9948-8
Springer

M. Newman (2010)
Networks: An Introduction
OUP Oxford