LQR control of complex networks

Roberto Galizia

National University of Ireland Galway

R.GALIZIA1@nuigalway.ie

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Overview

LQR optimal control

Let us consider a continuous time linear system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned} \tag{1}$$

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with a given initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Let us define a cost functional

$$J = \int_0^{+\infty} \mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t) + 2\mathbf{x}^T(t)\mathbf{M}\mathbf{u}(t) \tag{2}$$

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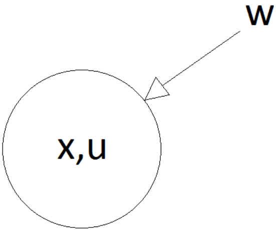
$$\mathbf{K} = \mathbf{R}^{-1}(\mathbf{B}^T\mathbf{P} + \mathbf{M}^T) \tag{4}$$

and the matrix P given by the positive solution of the Riccati equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - (\mathbf{P}\mathbf{B} + \mathbf{M})\mathbf{R}^{-1}(\mathbf{B}^T\mathbf{P} + \mathbf{M}^T) + \mathbf{Q} = \mathbf{0} \tag{5}$$

Network description

Consider a thermal system



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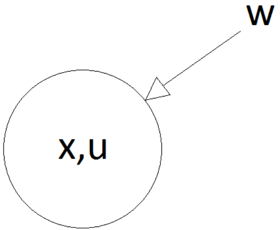

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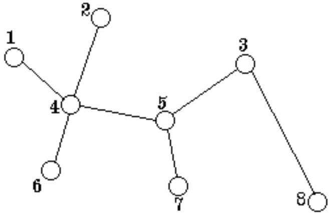
The dynamical model can then be found through heat balance.

$$\begin{cases} C\dot{x} = k(w - x) + u \\ y = x \end{cases} \tag{6}$$

where C is the thermal conductivity and k is the heat transfer coefficient.

Consider now an undirected complex network of N nodes.



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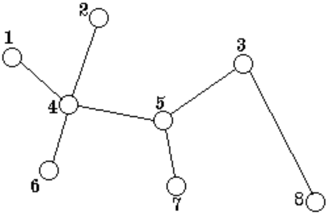

Defining the Adjacency matrix

$$\{\mathbf{A}_{dj}\}_{ij} = a_{ij} = \begin{cases} 1 & \text{if there is an edge between } i \text{ and } j \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

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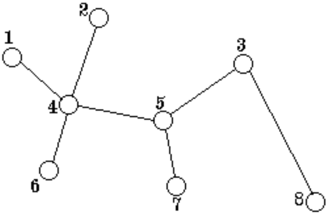

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the dynamics of a generic node is given by

$$\begin{cases} \dot{x}_i = \frac{k}{C} \sum_{j=1}^{N} a_{ij}(x_j - x_i) + \frac{1}{C} u_i \\ y_i = x_i \end{cases} \qquad i = 1, \cdots, N \tag{8}$$

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Introducing the Laplacian matrix

$$\{\mathbf{L}\}_{ij} = \begin{cases} k_i & \text{if } i = j \\ -1 & \text{if } i \neq j \wedge \text{ there is an edge between } i \text{ and } j \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

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$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \tag{10}$$

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the global differential equation that describes the dynamics of the network is

$$\begin{cases} \dot{\mathbf{x}} = -\frac{k}{C}\mathbf{L}\mathbf{x} + \frac{1}{C}\mathbf{I}\mathbf{u} \\ \mathbf{y} = \mathbf{x} \end{cases} \tag{11}$$

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Assuming that the desired state is constant

$$\mathbf{x}_d = \begin{bmatrix} x_{d_1} & \cdots & x_{d_N} \end{bmatrix}^T \tag{15}$$

we want the state $\mathbf{x}$ to converge to $\mathbf{x}_d$.

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then the choice

$$\tilde{\mathbf{u}} = -\mathbf{K}\epsilon \tag{19}$$

where $\mathbf{K}$ is calculated as in (4), minimizes the cost functional

$$J = \int_0^{+\infty} \epsilon^T\mathbf{Q}\epsilon + \tilde{\mathbf{u}}^T\mathbf{R}\tilde{\mathbf{u}} \tag{20}$$

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- The controller is not optimal anymore, but the computation can be spread all over the nodes.
- Adding/removing a node does not affect the whole network but only a small subset of nodes.

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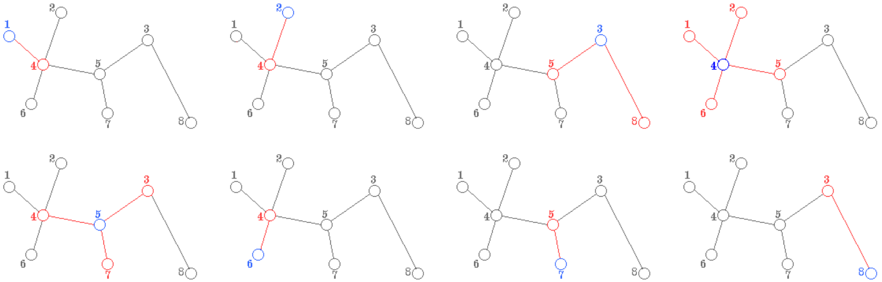

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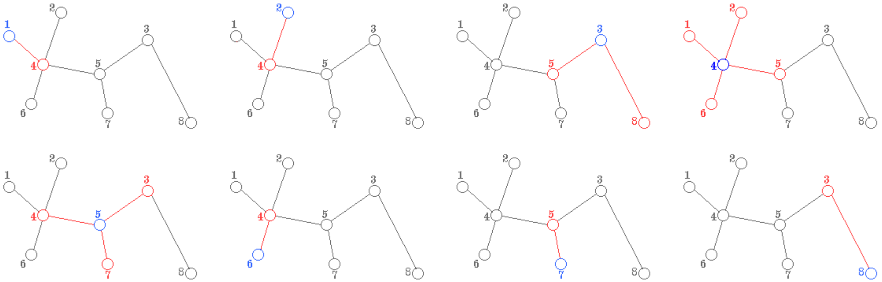

And the model for each node is

$$\dot{x}_i = \frac{k}{C} \sum_{j:x_j \in \mathbf{z}_i} (x_j - x_i) + \frac{1}{C} u_i \tag{21}$$

Decentralized control

Considering the Laplacian matrix L_i of the $i - th$ sub-network, the dynamics can be rewritten in matrix form as

$$\dot{\mathbf{z}}_i = -\frac{k}{C}\mathbf{L}_i\mathbf{z}_i + \frac{1}{C}\mathbf{b}_i u_i \tag{22}$$

where $\mathbf{b}_i$ is a vector with one only 1 at the position i and all 0 elsewhere.

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The control action can be chosen, as the LQR solution of the subnetwork,

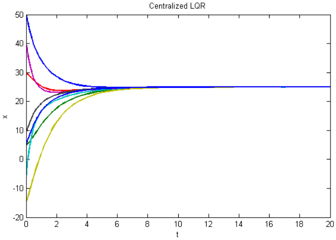
$$u_i = -\mathbf{B}_i^T(\mathbf{B}_i\mathbf{B}_i^T)^{-1}\mathbf{A}_i\mathbf{x}_{d_i} - \mathbf{K}_i\epsilon_i \tag{24}$$

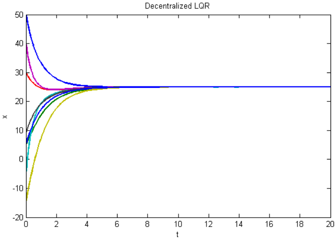
whit K_i calculated with (4), minimizing the cost functional

$$J_i = \int_0^{+\infty} \epsilon_i^T\mathbf{Q}\epsilon_i + ru_i^2 \tag{25}$$

Results and comparison

With $x_{d_i} = 25 \quad \forall i$, both the controllers complete the task





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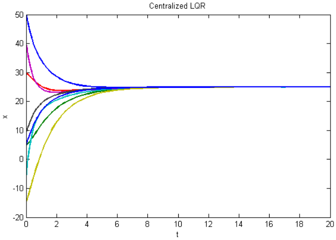
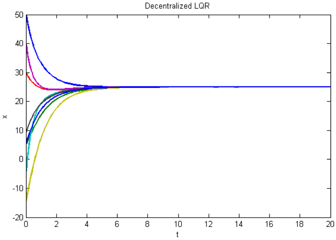
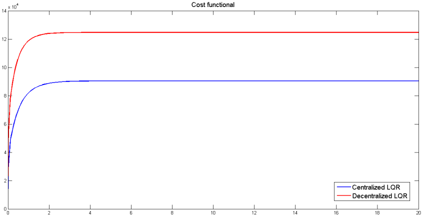




and as it was expected, the centralized control performs better



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- Induce structural changes by varying critical parameters
- Generalize the analysis to more complicated systems

References

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