The Inflation of Spherical Balloons

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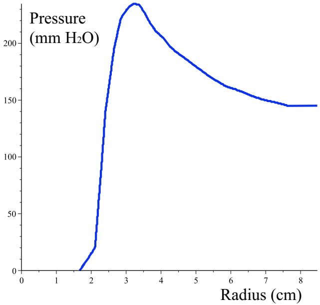
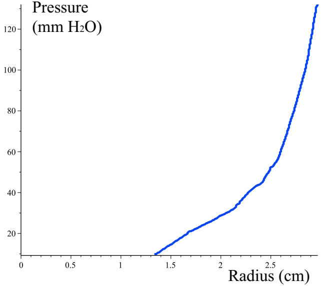
Introduction

We know from experience that inflating a rubber balloon is difficult in the beginning, but then it becomes easier. Finally, it becomes difficult again as the balloon approaches rupture. This is reflected in Osborne's 1909 experimental results (left).





He also tested a monkey bladder (right). In this case, the pressure-stretch curve exhibits monotonic increasing, exponential-like behaviour.

Modelling

- Spherical shell made of an incompressible isotropic hyperelastic material
- Shell is subject to internal pressure P.
- Assume purely radial deformation, $r = r(R)$, where r and R denote the deformed and undeformed radial distances, respectively.

We find that the deformation gradient, $\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X}$, is given by

$$\mathbf{F} = \mathrm{diag}(dr/dR, r/R, r/R), \tag{1}$$

where $\mathbf{x}$ and $\mathbf{X}$ are the deformed and undeformed positions of a material particle, respectively.

From this, the principal stretches (the square roots of the eigenvalues of $\mathbf{B} = \mathbf{F}\mathbf{F}^T$) are thus $\lambda_1 = dr/dR$ (*radial stretch*) and $\lambda_2 = \lambda_3 = r/R \equiv \lambda$ (*circumferential stretch*).

Using the incompressibility condition, $\det \mathbf{F} = \lambda_1\lambda_2\lambda_3 = 1$, we have

$$\frac{dr}{dR} = \frac{R^2}{r^2}. \tag{2}$$

Letting A, B and a, b denote the inner radius and outer radius of the shell in the reference and current configurations, respectively, and solving (2) eventually leads to

$$1 - {\lambda_a}^3 = \frac{R^3}{A^3}(1 - \lambda^3) = \frac{B^3}{A^3}(1 - {\lambda_b}^3), \tag{3}$$

where $\lambda_a = a/A$ and $\lambda_b = b/B$.

By spherical symmetry, the only non-zero components of the Cauchy stress tensor $\mathbf{T}$ are $t_1 = T_{11}$ (*radial stress*) and $t_2 = T_{22} = T_{33}$ (*hoop stress*).

For incompressible isotropic hyperelastic materials, these are given by given by

$$t_1 = \lambda_1 \frac{\partial W}{\partial \lambda_1} - p \quad \text{and} \quad t_2 = \lambda_2 \frac{\partial W}{\partial \lambda_2} - p, \tag{4}$$

where $W = W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda^{-2}, \lambda, \lambda)$ is the strain energy density function and p is a an arbitrary scalar.

For mechanical equilibrium, the equation of motion is

$$\mathrm{div}\ \mathbf{T} = \mathbf{0}, \tag{5}$$

where div denotes the divergence operator in the current configuration

The only non-trivial component of (5) is

$$\frac{dt_1}{dr} = \frac{2}{r}(t_2 - t_1). \tag{6}$$

Introducing the auxiliary function $\hat{W}(\lambda) = W(\lambda^{-2}, \lambda, \lambda)$ and performing some manipulations, we find

$$\frac{dt_1}{d\lambda} = \frac{\hat{W}'(\lambda)}{1 - \lambda^3}. \tag{7}$$

Because the shell is subject to internal pressure P, the boundary conditions are $t_1(\lambda_a) = -P$ and $t_1(\lambda_b) = 0$. Integrating (7) and imposing the boundary conditions, we find that

$$t_1(\lambda) = \int_{\lambda_b}^{\lambda} \frac{\hat{W}'(s)}{1 - s^3}\mathrm{d}s \quad \text{and} \quad P = \int_{\lambda_a}^{\lambda_b} \frac{\hat{W}'(\lambda)}{1 - \lambda^3}\mathrm{d}\lambda, \tag{8}$$

where s is a dummy variable.

Now, introducing the *thickness parameter* $\delta = (B - A)/A$ and noting from (3) that

$$\lambda_b = \left(1 - \frac{1 - \lambda_a^3}{(1 + \delta)^3}\right), \tag{9}$$

we can expand P in terms of δ to find

$$P = \delta \frac{\hat{W}'(\lambda)}{\lambda^2} + \frac{\delta^2}{2\lambda^4} \left[\frac{\lambda^3 - 2}{\lambda} \hat{W}'(\lambda) - (\lambda^3 - 1) \hat{W}''(\lambda) \right] + O(\delta^3). \tag{10}$$

Hence, for thin shells, P can be approximated by

$$P = \delta \frac{\hat{W}'(\lambda)}{\lambda^2}. \tag{11}$$

Laplace's Law

Also, expanding P/t_2 to first order in δ leads to

$$\frac{P}{t_2} = \frac{2}{\lambda^3}\delta. \tag{12}$$

The hoop stress is equal to the surface tension T divided by the deformed thickness of the shell. Combining both of these facts, we recover the classical membrane relation:

$$\boxed{T = \frac{Pr}{2},} \tag{13}$$

where r is the radius of the shell. This is sometimes called **Laplace's Law**. Similarly, it can be shown for a cylinder that

$$\boxed{T = Pr}. \tag{14}$$

Models

We consider the Mooney strain energy density function:

$$W_n = c_1(\lambda_1^n + \lambda_2^n + \lambda_3^n - 3) + c_2(\lambda_1^n\lambda_2^n + \lambda_2^n\lambda_3^n + \lambda_3^n\lambda_1^n - 3), \tag{15}$$

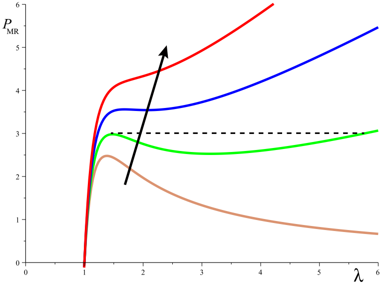
and the Gent-Gent model

$$W_{\mathrm{GG}} = -c_1 J_m \ln\left(1 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3}{J_m}\right) + c_2 \ln\left(\frac{\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2}{3}\right), \tag{16}$$

where n,c_1, c_2,J_m are positive constants. The Mooney-Rivlin model is given by (15) with $n = 2$. The Gent model is given by (16) with $c_2 = 0$.

Pressure-stretch curves for the Mooney-Rivlin model

Pressure-stretch curves (from (11)) for various values of the parameter $c' = c_2/c_1$



Three types of behaviour may occur:

- A monotonic increasing curve
- A limit-point instability, i.e. a maximum in the curve
- An inflation-jump instability, i.e. a sudden jump in radius for a small increase in pressure (see green curve)

Based on experimental data, the latter two are required to model rubber.

Instability analysis for the Mooney model

We found explicit conditions on the parameters for the shell to have a limit-point instability by solving the equations

$$\mathrm{d}P/\mathrm{d}\lambda = 0, \quad \mathrm{d}^2P/\mathrm{d}\lambda^2 = 0. \tag{17}$$

These conditions are $1.5 < n < 3$ and $c' < c'_{\mathrm{cr}}$ with

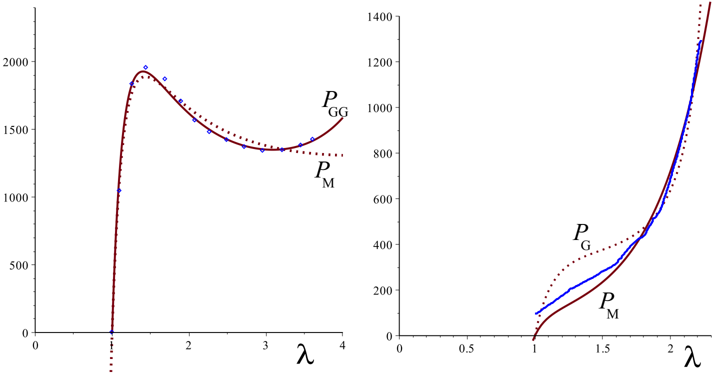
$$c'_{\mathrm{cr}} = \frac{(-2n-3)\left(\frac{-7n^2-3n\sqrt{5n^2-9}+9}{2n^2-9n+9}\right)^{-1/3} - (n-3)\left(\frac{-7n^2-3n\sqrt{5n^2-9}+9}{2n^2-9n+9}\right)^{2/3}}{(2n-3)\left(\frac{-7n^2-3n\sqrt{5n^2-9}+9}{2n^2-9n+9}\right) + n + 3},$$

and the corresponding critical value of the circumferential stretch:

$$\lambda_{\mathrm{cr}} = \left(-\frac{7\,n^2 + 3\,\sqrt{5\,n^4 - 9\,n^2} - 9}{2\,n^2 - 9\,n + 9}\right)^{\frac{1}{3n}}.$$

Curve fitting

We also fitted the theoretical model to the experimental data.



The Gent model provided the best fit for the rubber balloon, while the Mooney model provided the best fit for the bladder. However, the bladder is highly anisotropic so this model may be inadequate.

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- R. Mangan, M. Destrade, Gent models for the inflation of spherical balloons, International Journal of Non-Linear Mechanics, 68 (2015), 52–58.
- D.R. Merritt, F. Weinhaus, The pressure curve for a rubber balloon, American Journal of Physics 46 (1978) 976–977.