Degrees of Freedom in Rigidity Theory

Qays Shakir[1,2]

1. Natiaonal University of Ireland, Galway
2. Middle Techincal University

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Bar-joint Frameworks

Framework

A framework $\mathcal{F}$ in $\mathbb{R}^d$ is a pair (G, P) where $G = (V, E)$ is a graph and P is a map (usually called realisation or configuration)

$$P : V \to \mathbb{R}^d \text{ where } P(i) = p_i$$

such that $p_i \neq p_j$ whenever $ij \in E$.

An intuitive view of degrees of freedom

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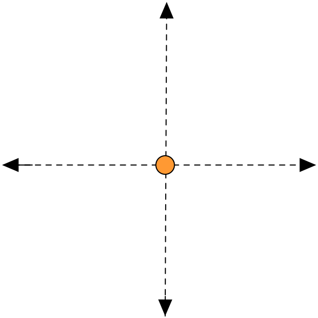
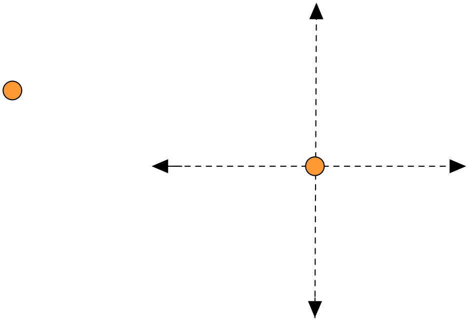
So the degree of freedom of a point in a plane is 2.



An intuitive view of degrees of freedom

Consider two points in a plane. The total number of degree of freedom is 4.



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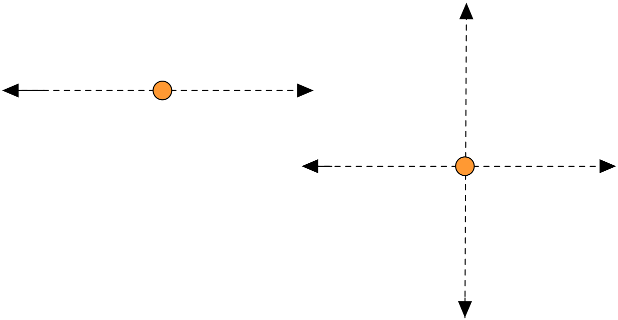

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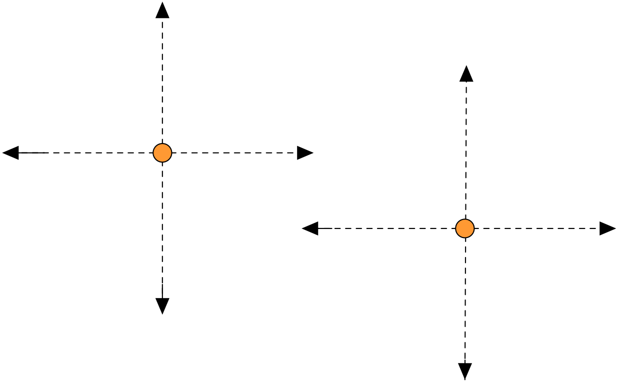

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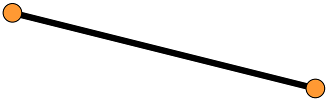
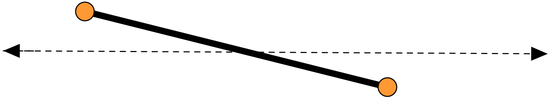


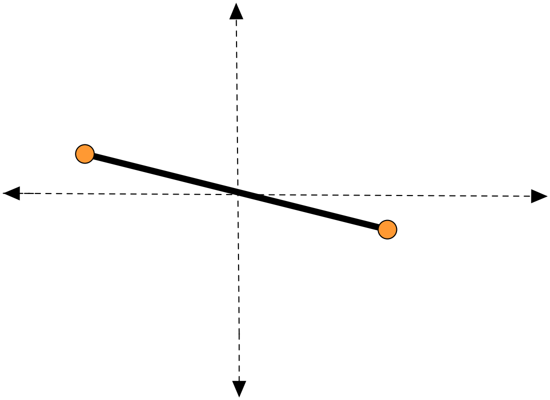
An intuitive view of degrees of freedom

The bar and the two joints (points) can move horizontally



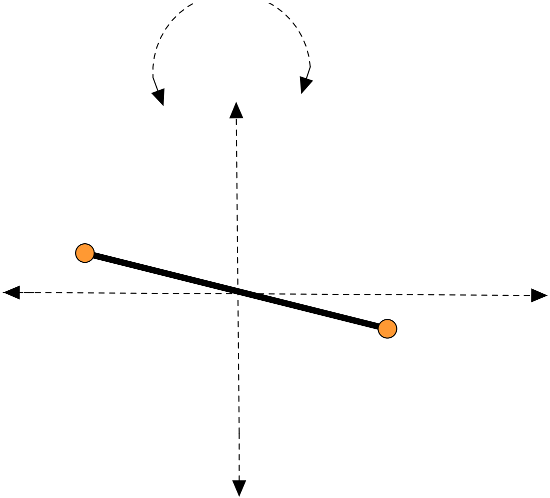
An intuitive view of degrees of freedom

The bar and the two joints (points) can move horizontally, vertically



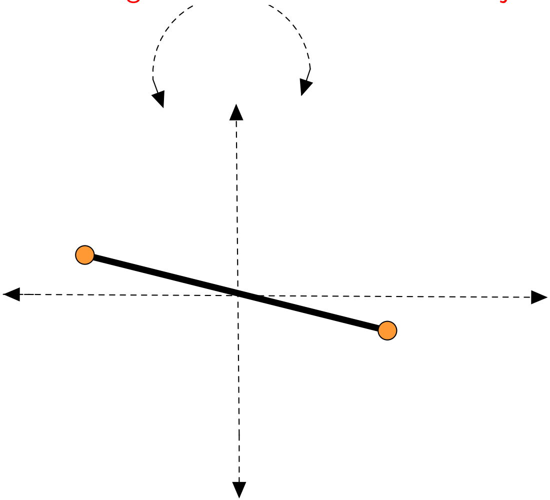
An intuitive view of degrees of freedom

The bar and the two joints (points) can move horizontally, vertically and it can be rotated.



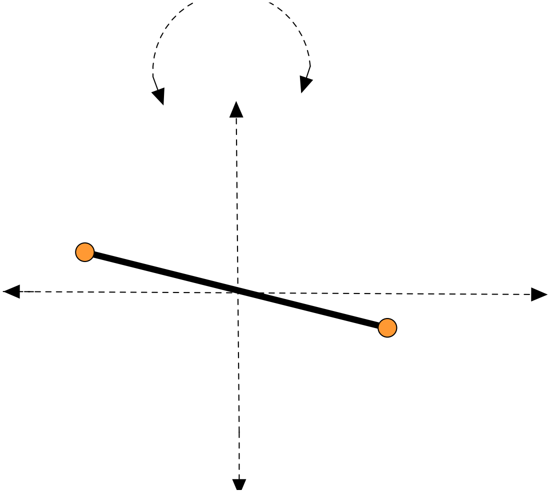
An intuitive view of degrees of freedom

The bar and the two joints (points) can move horizontally, vertically and it can be rotated. So the degree of freedom of the whole object is 3



An intuitive view of degrees of freedom

The bar and the two joints (points) can move horizontally, vertically and it can be rotated. So the degree of freedom of the whole object (single bar) is 3



Infinitesimal Rigidity

Infinitesimal motion

Let $\mathcal{F} = (G, p)$ be a framework in $\mathbb{R}^d$. An infinitesimal motion of $\mathcal{F} = (G, p)$ is a function $q : V \longrightarrow \mathbb{R}^d$ such that

$$\langle p(v_i) - p(v_j)\rangle . \langle q(v_i) - q(v_j)\rangle = o \text{ for all edges } \{v_i, v_j\} \in E(G)$$

Infinitesimal rigidity

A framework $\mathcal{F} = (G, p)$ is infinitesimally rigid if it is not admit any infinitesimal motion

Rigidity Matrix

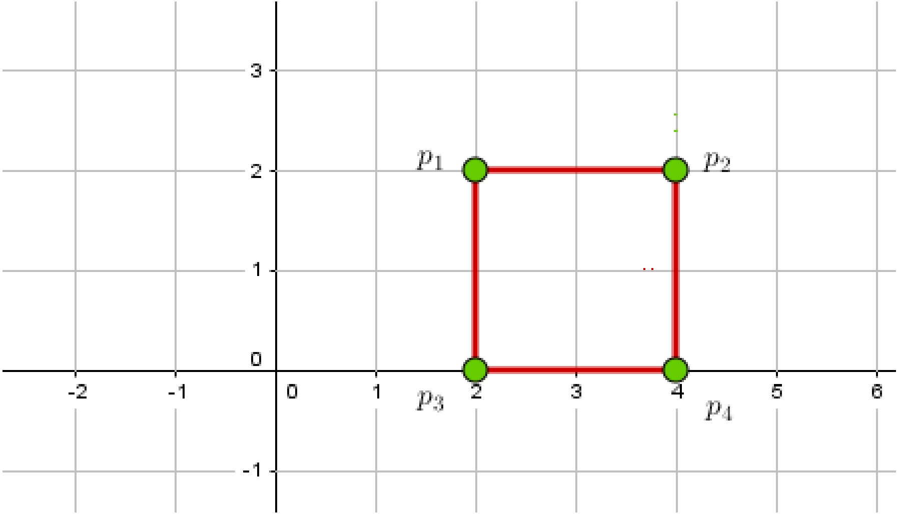
Let framework $\mathcal{F} = (G, p)$ be a d-dimensional framework. The rigidity matrix $R(G, p)$ of $\mathcal{F}$ is a $|E| \times d|V|$ matrix whose rows are indexed by the edges of G and whose columns are indexed by the vertices of G such that :

$$R = \begin{array}{c} \\ \{i,j\} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & \dots & i & \dots & j & \dots & n \end{array} \\ \begin{pmatrix} \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & (p_i - p_j) & \dots & (p_j - p_i) & \dots & 0 \\ \vdots & \ddots & \vdots & \dots & \vdots & \ddots & \vdots \end{pmatrix} \end{array}$$

Theorem

A framework $\mathcal{F} = (G, p)$ is infinitesimally rigid in $\mathbb{R}^d$ with $n = |V| \geq d$ if and and only if $rank(R) = nd - \frac{d(d+1)}{2}$

Rigidity matrix



Degree of freedom via the rigidity Matrix

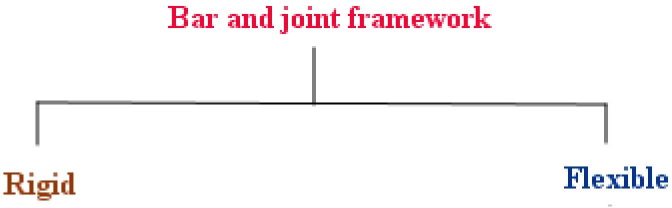
Total degrees of freedom

Total degrees of freedom of $\mathcal{F}=$
dim(the space of the solution space of $Rq^t = 0$)

Internal degrees of freedom

Internal degrees of freedom of the framework $=$
dim(Space of infinitesimal motions) $-$ dim(Space of trivial motion)

Degrees of freedom in three spaces



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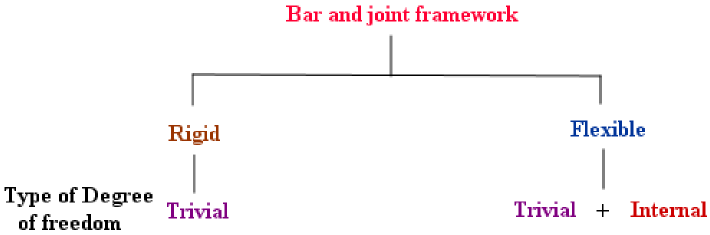

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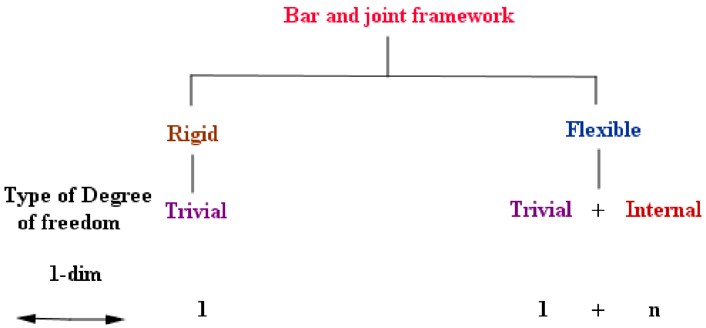

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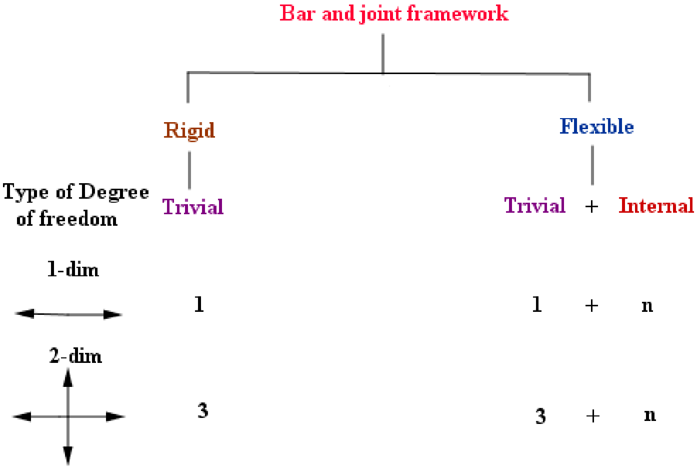

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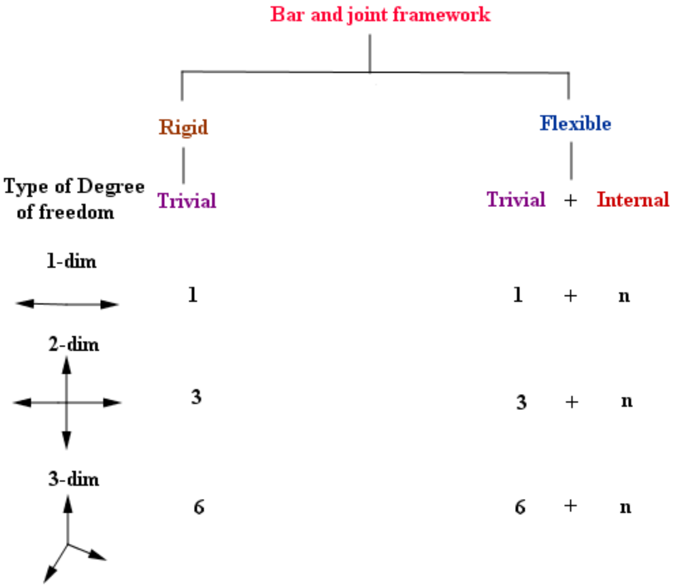
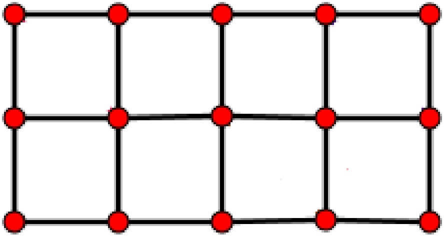


Grid bracing problem

An $m \times n$ grid is a framework $\mathcal{F} = (G, p)$ where $P : V \to \mathbb{R}^2$.



Degree of freedom of a grid

A degree of freedom of a grid is the number of braces required to rigidify it

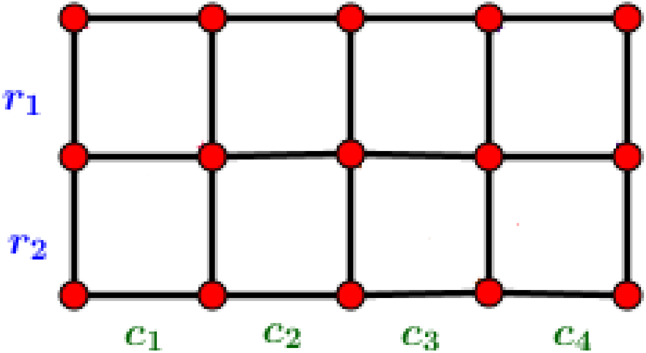
Degree of freedom of a grid

The degree of freedom of a grid is $m + n - 1$

Grid bracing problem; The brace graph

The brace graph contains a vertex for each row and each column of the cell grid. The vertices will encode the bracing of the unit grid as follows: If the cell in row r_i and column c_j is braced, the vertices of the brace graph labeled r_i and c_j are joint by an edge.

Grid bracing problem



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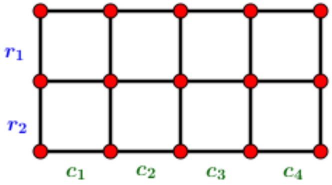

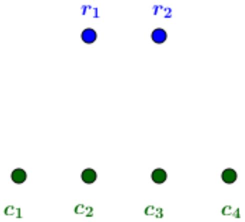

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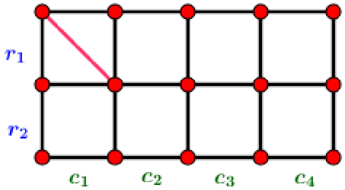

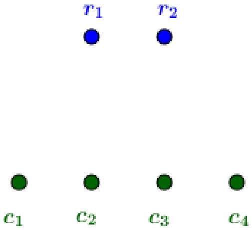

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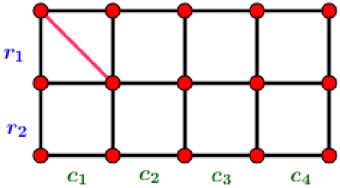

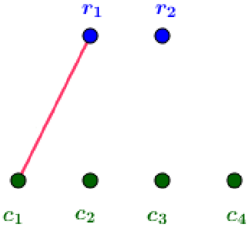

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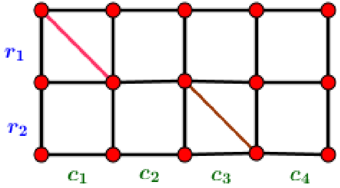

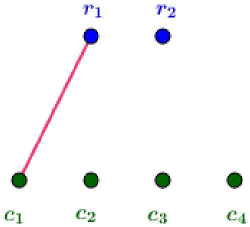

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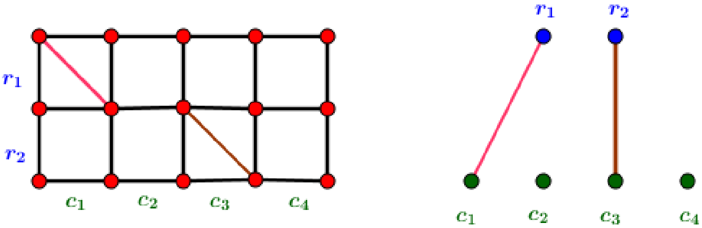

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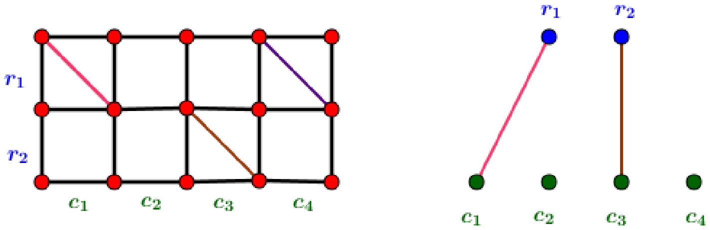

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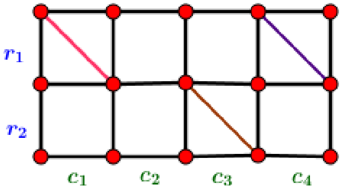
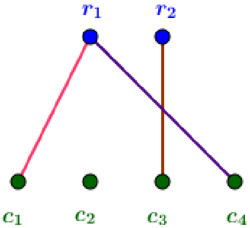
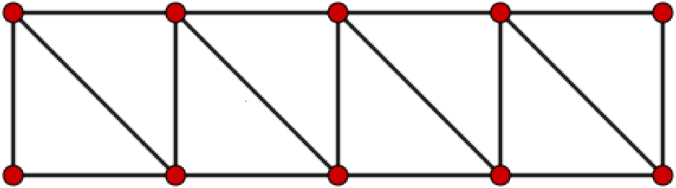




Grid bracing problem; removing a brace



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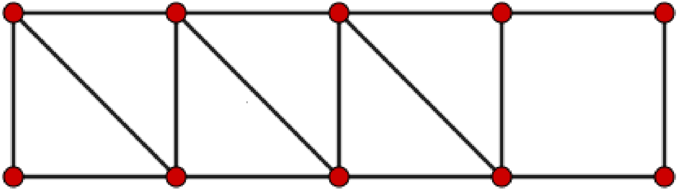

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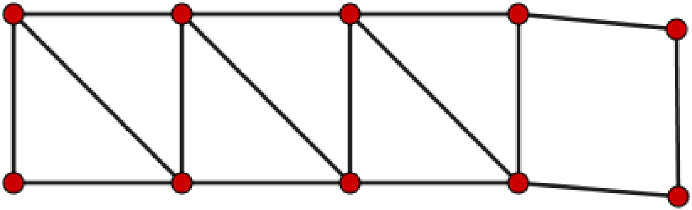

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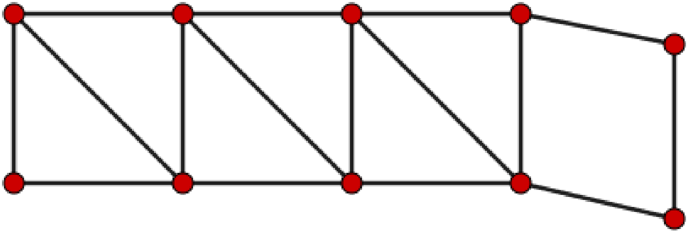

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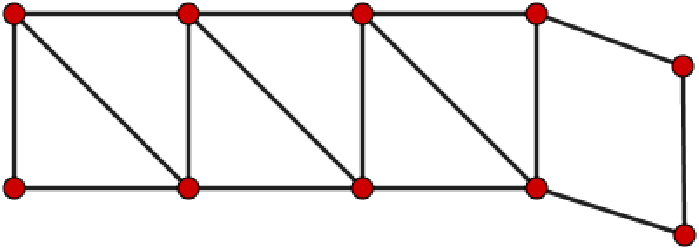

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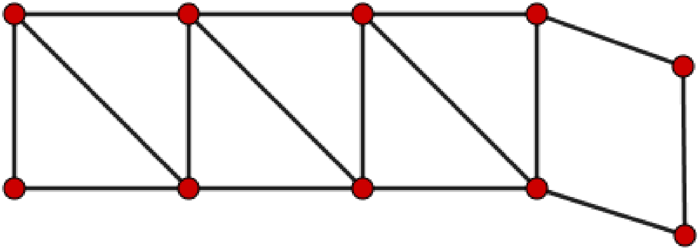

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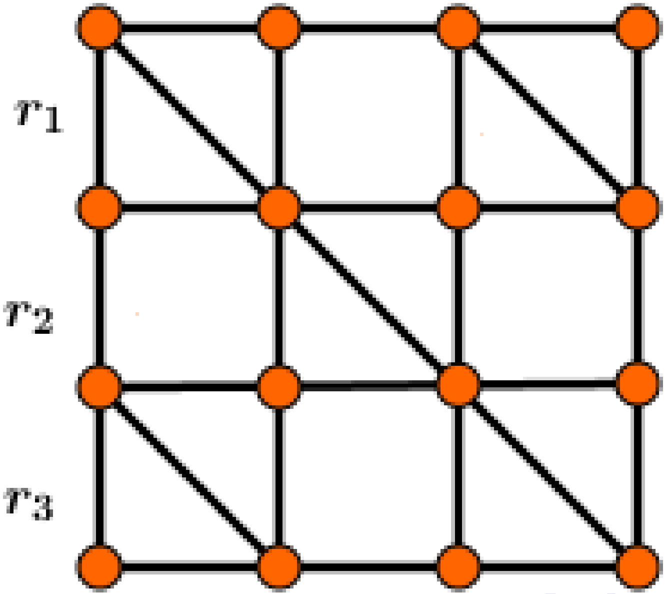

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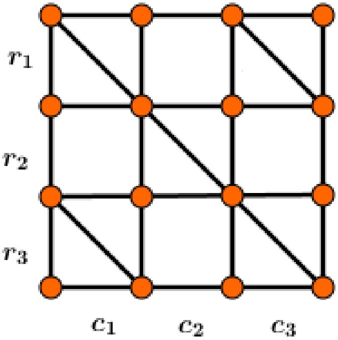

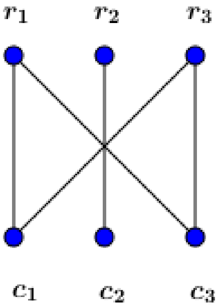

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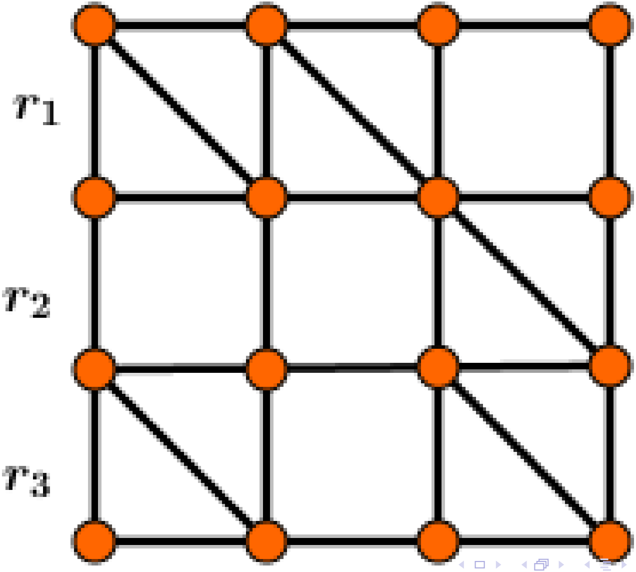

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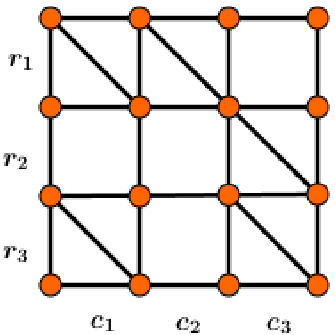

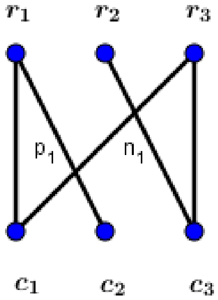

References

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