Membrane Mechanics

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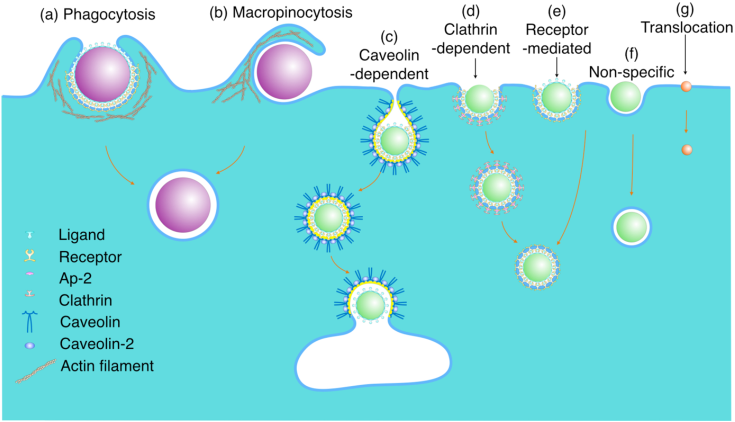
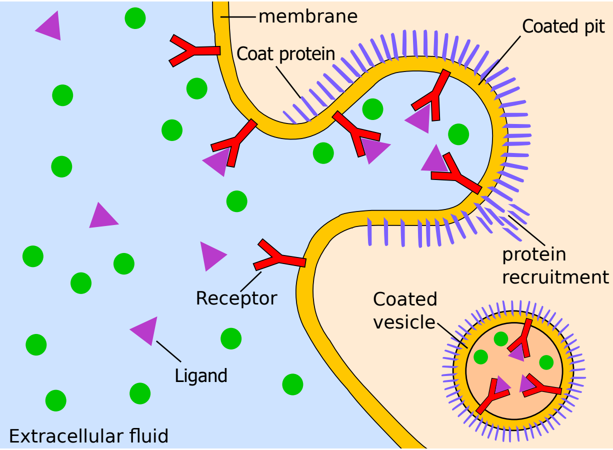
Endocytosis



Zhang et al., *Physical Principles of Nanoparticle Cellular Endocytosis*, ACS Nano, 9, 8655.

Receptor-mediated Endocytosis



- Ligands on NP surface bind to receptors on cell membrane
- Signalling from receptors leads to protein recruitment and pit formation
- Pinches off to form vesicle

Membrane Modelling

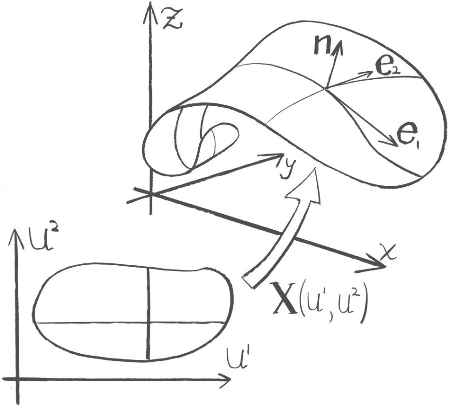
- Separation of lengthscales: lateral dimensions greatly exceed thickness.
- Parameterisation of surface: 2D coordinate system $\{u^1, u^2\}$ with map

$$\mathbf{r} = \mathbf{X}(u^1, u^2) = \begin{pmatrix} X(u^1, u^2) \\ Y(u^1, u^2) \\ Z(u^1, u^2) \end{pmatrix}$$

- Tangent vectors

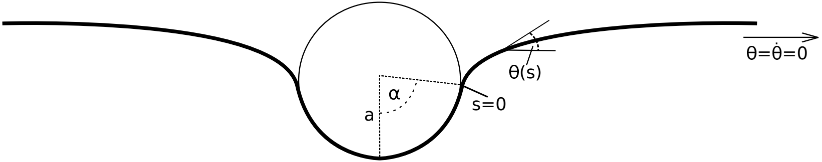
$$\mathbf{e}_a = \frac{\partial \mathbf{X}}{\partial u^a} = \partial_a \mathbf{X}, a \in \{1, 2\}$$

span local tangent plane to surface



A Minimal Model

Consider an infinite rod, bending rigidity κ, in contact with a disc of radius a, contact energy w per unit length.



Require membrane to be flat far away from disc: BCs are

$$\lim_{s\to\infty} \theta(s) = \lim_{s\to\infty} \dot{\theta}(s) = 0. \quad (1)$$

Amount of stretching in bound part of membrane: (length of portion of rod in flat configuration) - (length in the wrapped configuration) $= 2\alpha a - 2a\sin(\alpha)$. This gives the contact energy

$$E_{contact} = 2\alpha a\left(\frac{\kappa}{2a^2} - w\right) + 2a\sigma(\alpha - \sin(\alpha)).$$

A Minimal Model

Amount of stretching in free part of the membrane is difference between length of curved piece & flat piece, $ds - dx$, so contribution of stretching to energy of free membrane is

$$\int_{s=0}^{s=\infty} \sigma(ds - dx) = \int_0^\infty \sigma\left(1 - \frac{dx}{ds}\right) ds = \int_0^\infty \sigma(1 - \cos(\theta))ds,$$

and combining this with usual expression for bending energy gives energy of free portion

$$E_{free} = 2\int_{s=0}^{s=\infty} \left[\sigma(1 - \cos(\theta)) + \frac{1}{2}\kappa\left(\frac{d\theta}{ds}\right)^2\right] ds. \qquad (2)$$

A Minimal Model

Thus the total energy is

$$\begin{aligned} E_{total} =& 2\alpha a \left(\frac{\kappa}{2a^2} - w \right) + \\ & 2a\sigma(\alpha - \sin(\alpha)) + 2\int_{s=0}^{s=\infty} \left[\sigma(1 - \cos(\theta)) + \frac{1}{2}\kappa \left(\frac{d\theta}{ds} \right)^2 \right] ds \\ =& \int_{s=0}^{s=\infty} (\kappa\dot{\theta}^2 + 2\sigma(1 - \cos(\theta)) + C\dot{\theta}) ds, \end{aligned}$$

for constant C.

Define the Lagrangian

$$L(\theta, \dot{\theta}) = \kappa\dot{\theta}^2 + 2\sigma(1 - \cos(\theta)) + C\dot{\theta},$$

from which we obtain the Euler-Lagrange equation

$$\ddot{\theta} - \frac{\sigma}{\kappa}\sin(\theta) = 0. \tag{3}$$

Solving this subject to the boundary conditions gives

$$\int \frac{d\theta}{\sqrt{1 - \cos(\theta)}} = \sqrt{\frac{2\sigma}{\kappa}}s + E, \tag{4}$$

where E is an arbitrary constant.

A Minimal Model

Integrating and imposing $\theta(s=0)=\alpha$ gives

$$\theta(s) = 2\cos^{-1}(\tanh(\tanh^{-1}(\cos(\alpha/2)) - \sqrt{\sigma/\kappa}s)),$$

and the profile of the free part of the membrane can now be calculated using

$$(x(s), y(s)) = \left(a\sin(\alpha) + \int_0^s \cos(\theta(s))ds, -a\cos(\alpha) + \int_0^s \sin(\theta(s))ds\right).$$

where we have specified the coordinates of the first point of contact for a given degree of wrapping α by

$$(x(0), y(0)) = (a\sin(\alpha), -a\cos(\alpha)).$$

A Minimal Model

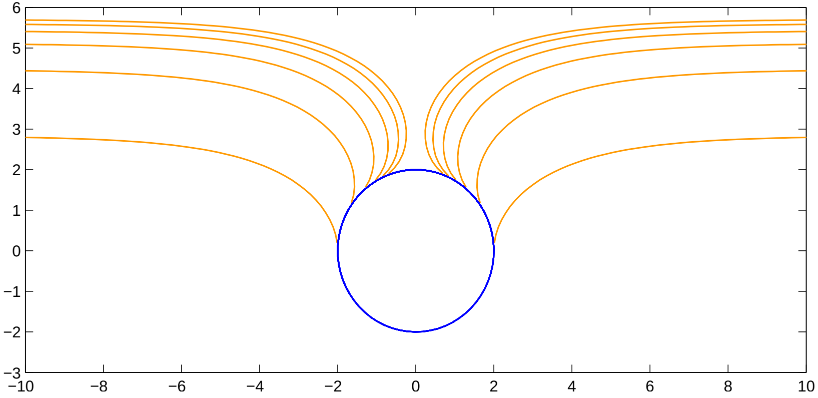


Figure: Membrane profiles for various values of wrapping angle α

References

Zhang et al.
Physical Principles of Nanoparticle Cellular Endocytosis.
ACS Nano, **9**, 8655.

Markus Deserno.
Fluid lipid membranes: From differential geometry to curvature stresses.
Chemistry and Physics of Lipids, **185** , 11–45.