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Genus g Zhu Recursion for Vertex Operator Algebras

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April 20, 2018

Introduction

In this talk, we will outline recent work done on general genus Zhu recursion for vertex operator algebras.

Vertex Operator Algebras

We begin with a brief recap of vertex operator algebras (VOAs). A vertex operator algebra is a quadruple $(V, Y(,), \mathbf{1}, \omega)$ consisting of the following data:

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- A *vacuum* vector $\mathbf{1} \in V$
- A Virasoro vector $\omega \in V$

Vertex Operator Algebras

This data obeys the following axioms:

- For all u, v in V, there exists an integer N such that:

  $$(z-w)^N[Y(u,z),Y(v,w)]=0$$

  where $[,]$ is the commutator defined by:

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- $Y(\mathbf{1}, z) = Id_V$
- $Y(u,z)\mathbf{1} = u + O(z)$

VOAs continued

- $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ where the $L(n)$ operators satisfy the Virasoro Lie algebra:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m,-n}c$$

where c is a parameter known as the *central charge*.

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- The $L(0)$ operator induces a grading on V, i.e.

$$V = \bigoplus_{n\in\mathbb{N}} V_n$$

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$$\{v \in V : L(0)v = nv, n \in \mathbb{N}\}$$

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- $Y(L(-1)v, z) = \frac{d}{dz}Y(v, z)$

Genus Zero Zhu Recursion and Higher...

In previous talks, we have mostly examined genus two Zhu recursion, building on genus one data.

The aim of this project is to find a formula for all genera, building on genus zero data using the *canonical formalism*.

The Canonical Formalism

Define the indexing sets

$$\mathcal{I} = \{-1, \ldots, -g, 1, \ldots, g\}, \quad \mathcal{I}_+ = \{1, 2, \ldots, g\}$$

To construct a genus g surface, we begin by excising $2g$ discs on the sphere $\mathcal{S}^{(0)}$. Let Let $\{Q_a\}$ be a set of $2g$ points on $\mathcal{S}^{(0)}$ for $a \in \mathcal{I}$. Let z_a denote a local coordinate in the neighbourhood of Q_a. We introduce g sewing parameters $\rho_a = \rho_{-a}$ and excise $2g$ disks

$$|z_a| < \frac{|\rho_a|}{r_{-a}}$$

for real $r_a > 0$ to form a sphere with $2g$ punctures. Denote this new surface by $\widehat{\mathcal{S}}^{(0)}$.

The Canonical Formalism

Define $2g$ annuli on $\widehat{\mathcal{S}}^{(0)}$ centred at Q_a by

$$\mathcal{A}_a = \{|\rho_a|/r_{-a} \leq |z_a| \leq r_a\}$$

The construction of the genus g surfaces is completed by identifying the annuli $\mathcal{A}_a$ and $\mathcal{A}_{-a}$ using the sewing relation

$$z_a z_{-a} = \rho_a; a \in \mathcal{I}_+$$

with the stipulation that the parameters r_a are sufficiently small so as to prevent intersection of distinct annuli.

Taking the limit $\rho_a \to 0$ for all $a \in \mathcal{I}_+$, we see that the genus g surface $\mathcal{S}^{(g)}$ degenerates to the Riemann sphere $\mathcal{S}^{(0)}$.

The Canonical Formalism

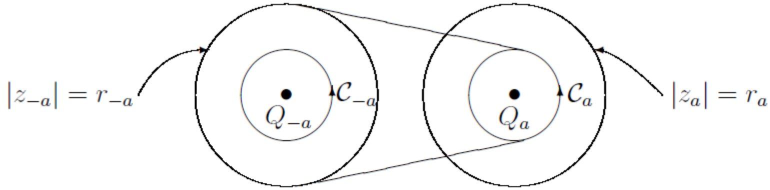

Fig 1. Sewing of annuli.

Genus Zero Zhu Recursion

The genus zero n-point function is defined for states $v_k \in V$, $k = 1, \ldots n$ by

$$Z_V^{(0)}(\boldsymbol{v}, \boldsymbol{y}) = \langle \mathbf{1}, \boldsymbol{Y}(\boldsymbol{v}, \boldsymbol{y})\mathbf{1} \rangle$$

where $\boldsymbol{Y}(\boldsymbol{v}, \boldsymbol{y}) = Y(v_1, y_1)Y(v_2, y_2) \ldots Y(v_n, y_n)$ and $\mathbf{1}$ denotes the vacuum vector.

Genus Zero Zhu Recursion

We find that the genus zero n-point function obeys the following recursion formula:

Theorem (Genus Zero Zhu Recursion)

For u quasiprimary of weight N, the genus zero $(n+1)$-point function obeys the following Zhu recursion formula

$$Z_V^{(0)}(u,x;\boldsymbol{v},\boldsymbol{y}) = \sum_{k=1}^{n}\sum_{j\geq 0}\partial^{(0,j)}\zeta_N(x,y_k)Z_V^{(0)}(\ldots;u(j)v_k,y_k;\ldots). \tag{1}$$

Genus Zero Zhu Recursion

where $\partial^{(i,j)} f(x,y) = \partial_x^{(i)} \partial_y^{(j)} f(x,y)$ and $\partial^{(i)} = \frac{1}{i!}\partial^i$, with $\zeta_N(x,y)$ given by

$$\zeta_N(x,y) = \frac{1}{x-y} + \sum_{\ell=0}^{2N-2} f_\ell(x) y^\ell,$$

where $f_\ell(x)$ is any Laurent series in x. We have found that we have a large degree of latitude in the choice of $f_\ell(x)$, however some choices are more "interesting" than others.

We now want to extend this result to any genus.

General Genus n-Point Functions

Let $\boldsymbol{b}_+$ denote an element of a basis $\{b_a\}$ for $V^{\otimes g}$, i.e. g copies of V, with $a \in \mathcal{I}_+$, and let $\overline{b}_a$ denote the dual vector with respect to the ρ_a-dependent Li-Z metric $\langle \cdot, \cdot \rangle_a$. Now for $\overline{b}_a \in V_{n_a}$, define for $a \in \mathcal{I}_+$

$$b_{-a} = \overline{b}_a.$$

Now we will consider Zhu recursion for genus g n-point functions. The genus g n-point function is given by:

$$Z_V^{(g)}(\boldsymbol{v}, \boldsymbol{y}) = \sum_{\boldsymbol{b}_+} Z_V^{(0)}(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{b}, \boldsymbol{w}),$$

where

$$Z_V^{(0)}(\boldsymbol{v}, \boldsymbol{y}; \boldsymbol{b}, \boldsymbol{w}) = Z_V^{(0)}(v_1, y_1; \ldots; v_n, y_n; b_{-1}, w_{-1}; \ldots; b_g, w_g).$$

General Genus Zhu Recursion

We eventually find that

Theorem

Genus g Zhu Recursion The genus g n-point function for a quasiprimary vector u of weight $\mathrm{wt}(u) = N$ *inserted at* $x \in \mathcal{S}^{(g)}$ *and general vectors* $v_1, v_2 \ldots, v_n$ *inserted at* $y_1, y_2 \ldots, y_n \in \mathcal{S}^{(g)}$ *respectively, obeys the recursive identity*

$$Z_V^{(g)}(u, x; \boldsymbol{v}, \boldsymbol{y}) = \phi^{(g)}(x) X^{\Pi} + \sum_{k=1}^{n} \sum_{j \geq 0} \partial^{(0,j)} \zeta_N^{(g)}(x, y_k) Z_V^{(g)}(\ldots; u(j)v_k, y_k; \ldots). \quad (2)$$

where $\phi^{(g)}(x)$ is a doubly indexed row vector given by

$$\phi^{(g)}(x) = \left(A(x) + \widetilde{A}(x) \left(I - \widetilde{R} \right)^{-1} (R + \Lambda\Gamma) \right) \Pi,$$

General Genus Zhu Recursion

$X^{\Pi} = (X^{\Pi}_a(m))$ is given by

$$X^{\Pi}_a(m) = \rho_a^{-\frac{m}{2}} \sum_{\boldsymbol{b}_+} Z^{(0)}_V(\ldots u(m)b_a, w_a; \ldots),$$

$m = 0, \ldots, 2N-2$. Lastly, $\zeta^{(g)}_N(x,y)$ is given by

$$\zeta^{(g)}_N(x,y) := \zeta_N(x,y) + \widetilde{A}(x)(I - \widetilde{R})^{-1}B(y),$$

where the following are all indexed by $a, b \in \mathcal{I}$, $m, n \geq 0$:

$$A_a(x, m) = \rho_a^{\frac{m}{2}} \partial^{(0,m)} \zeta_N(x, w_a), \quad \widetilde{A}(x) = A(x)\Delta,$$

with

$$\Delta_{ab}(m, n) = \delta_{m,n+K+1}\delta_{ab},$$

similarly,

$$\Gamma_{ab}(m, n) = \delta_{m,-n+K}\delta_{a,-b},$$

and

$$R_{ab}(m, n) = \begin{cases} (-1)^N \rho_a^{\frac{m+1}{2}} \rho_b^{\frac{n}{2}} \partial^{(m,n)} \zeta_N(w_{-a}, w_b), & a \neq -b, \\ 0, & a = -b. \end{cases}$$

with $\widetilde{R} = R\Delta$ and $\left(I - \widetilde{R}\right)^{-1} = \sum_{k \geq 0} \widetilde{R}^k$.

Genus g Objects

Lastly,

$$\Lambda_{ab}(m,n) = (-1)^N \rho_a^{\frac{i+j+1}{2}} \partial_{w_{-a}}^{(j)} \mathcal{F}_i(w_{-a})\delta_{ab},$$

where $\mathcal{F}_i$ depends on the choice of $\{f_\ell(x)\}$ in $\zeta_N(x,y)$, and

$$B_a(y;m) = (-1)^N \rho_a^{\frac{m+1}{2}} \partial^{(m,0)} \zeta_N(w_{-a},y).$$

$\zeta_N^{(g)}(x, y)$ as a Poincaré Sum

We have found that $\zeta_N^{(g)}(x, y)$ can be expressed as a Poincaré sum over the genus g Schottky group

$$\zeta_N^{(g)}(x, y)dx^N = \sum_{\gamma \in \Gamma_S^{(g)}} \zeta_N^{(0)}(\gamma x, y)d(\gamma x)^N$$

The Entries of $\phi^{(g)}(x)$

We believe that the vector $\phi^{(g)}(x)$ comprises a (non-independent) spanning set of differential forms on the genus g Riemann surface.

We are currently working on constructing a normalised basis from these entries with dimension complying with the Riemann-Roch Theorem.

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