Genus Two n-point Functions for VOAs I

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Introduction

In this talk we will discuss the idea of a genus one *n*-point function for a VOA, along with genus one Zhu recursion.

We will also motivate the notion of a genus two version of this function and the idea of Zhu recursion for such an object.

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- A Virasoro vector $\omega \in V$

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- $Y(u,z)\mathbf{1} = u + O(z)$

VOAs continued

- $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ where the $L(n)$ operators satisfy the Virasoro Lie algebra:

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- $Y(L(-1)v, z) = \frac{d}{dz}Y(v, z)$

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- has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$$

  where $q = exp(2\pi i\tau)$. This converges for $|q| < 1$ (i.e. $\Im(\tau) > 0$)

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where q is as before, B_k is a Bernoulli number and $\sigma_{k-1}(n)$ is the divisor function $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

Elliptic Functions

Following on from the E_k above we define:

$$P_n(z,\tau) = \frac{1}{z^n} + (-1)^n \sum_{k=n}^{\infty} \binom{k-1}{n-1} E_k(\tau) z^{k-n}$$

Note that there is no contribution from the odd k cases as then the E_k are trivial forms.

These functions obey the periodicities:

$$P_k(z + 2\pi i, \tau) = P_k(z,\tau)$$

$$P_k(z + 2\pi i\tau, \tau) = P_k(z,\tau) - \delta_{k1}$$

We now define an n-point function for a VOA by:

$$Z_V^{(1)}(v_1, z_1; \ldots; v_n, z_n; \tau)$$
$$= Tr(Y(q_1^{L(0)} v_1, q_1) \cdots Y(q_n^{L(0)} v_n, q_n) q^{L(0)-c/24})$$

where $q_i = \exp(z_i) = \sum_{n \geq 0} \frac{z_i^n}{n!}$ is a formal series in z_i.

Zhu Recursion

Zhu developed a recursion formula relating genus one n-point functions to $(n-1)$-point functions:

$$Z_V^{(1)}(v,z;v_1,z_1;\ldots;v_n,z_n;\tau)$$
$$= Tr_V\left(o(v)Y(q_1^{L(0)}v_1,q_1)\cdots Y(q_n^{L(0)}v_n,q_n)q^{L(0)-c/24}\right)$$
$$+\sum_{k=2}^{n}\sum_{j\geq 0}P_{1+j}(z-z_k,\tau)Z_V^{(1)}(v_1,z_1;\ldots;v[j]v_k,z_k;\ldots;v_n,z_n;\tau)$$

where $o(v)=v(wt(v)-1)$ and $v[j]$ is the coefficient of z^{-j-1} in $Y[v,z]=Y(q_z^{L(0)}v,q_z-1)$, with $q_z=\exp(z)$.

Genus Two n-Point Functions

One can also define genus two versions of n-point functions, building up from genus one data. This is implemented by a sewing scheme for Riemann surfaces, where two tori ("left" and "right") are attached (more on this in the next talk) to construct a double torus.

Figure: A genus two surface (double torus)

The definition of this function with $L+1$ states on the left torus and R states on the right is as follows (for $n = L + R + 1$):

$$Z_V^{(2)}(v, x; \boldsymbol{a_l}, \boldsymbol{x_l} | \boldsymbol{b_r}, \boldsymbol{y_r}; \tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_V^{(1)}(Y[v, x]\boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}]u, \tau_1) Z_V^{(1)}(\boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}]\overline{u}, \tau_2)$$

where $\boldsymbol{a_l}, \boldsymbol{x_l}$ denotes arguments $a_1, x_1, \ldots, a_L, x_L$, and likewise for the $\boldsymbol{b_r}$ terms.
Similarly, the bold vertex operator notation is used to streamline the products:

$$\boldsymbol{Y}[\boldsymbol{a_l}, \boldsymbol{x_l}] := Y[a_1, x_1] \cdots Y[x_L, x_L]$$

$$\boldsymbol{Y}[\boldsymbol{b_r}, \boldsymbol{y_r}] := Y[b_R, y_R] \cdots Y[b_1, y_1]$$

Genus Two n-Point Functions

The objective, then, is to find a genus two analogue of the Zhu recursion formula outlined above. Similar to how the $P_k(z,\tau)$ functions have periodicities on a torus, one expects to find a new class of objects with a natural interpretation on a double torus. More on this in the next talk.

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