# Nonsingular Entry Pattern Matrices

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# Entry pattern matrices

An entry pattern matrix (EPM for short) is a matrix in which:

- Each entry is an element of a specified set of independent indeterminates.
- Entries can be the same, but can not be a constant.

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**Example:** Let

$$A = \begin{bmatrix} x & y \\ z & x \end{bmatrix}, B = \begin{bmatrix} x+y & 0 \\ z & x \end{bmatrix}$$

Then $A$ is an entry pattern matrix with 3 indeterminates $\{x, y, z\}$ while $B$ is not.

# Nonsingular EPMs

A square EPM $A(x_1, \cdots, x_k)$ is said to be almost nonsingular over a field $\mathbb{F}$ (or $\mathbb{F}$-almost nonsingular) if $\det A(a_1, \cdots, a_k) \neq 0$ for all vector $(a_1, \cdots, a_k) \neq (constant)(1, \cdots, 1)$.

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**Example.** Let

$$A(x,y,z) = \begin{bmatrix} y & z & z & x & z \\ z & y & x & x & x \\ x & x & y & z & x \\ z & x & y & y & z \\ x & z & y & y & y \end{bmatrix}.$$

Then $A$ is almost nonsingular over $\mathbb{F}_3$ since $\det A(x,y,z) = 0 \mod 3$ precisely if $x = y = z$ but $A$ is not almost nonsingular over $\mathbb{F}_5$ since $\det A(0,1,2) = 0$

# The maximum possible number of indeterminates $\tau_{\mathbb{F}}(n)$

Given a field $\mathbb{F}$ and an integer $n$, denote $\tau_{\mathbb{F}}(n)$ to be the maximum possible number of indeterminates appearing in an $n \times n$ EPM $A$ so that $A$ is almost nonsingular over $\mathbb{F}$. **Determine $\tau_{\mathbb{F}}(n)$.**

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**Example:**

- $\tau_{\mathbb{F}_2}(2) = 2 : A(x, y) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$
- $\tau_{\mathbb{F}_2}(3) = \tau_{\mathbb{F}_2}(2) = 2 : A(x, y) = \begin{bmatrix} x & x & y \\ y & x & y \\ y & y & x \end{bmatrix}$
- $\tau_{\mathbb{C}}(n) = 2$ for every $n$ since $\mathbb{C}$ is algebraic closed.

# Bounds of $\tau_{\mathbb{F}}(n)$

- $\tau_{\mathbb{F}}(n) \le n : A = x_1 A_1 + \cdots + x_k A_k$, each $A_i$ has at least $n$ non-zero entries and no two of them have non-zero entry in the same position.
- $\tau_{\mathbb{F}}(n) \ge 2$ for $n \ge 4$:

$$T_4(x,y) = \begin{bmatrix} x & y & x & x \\ y & y & x & y \\ x & x & x & y \\ x & y & y & y \end{bmatrix}, \det T_4(x,y) = (x-y)^4$$

- $\tau_{\mathbb{R}}(n) = 2$ if $n$ is odd: $\det A(x,0,1)$ is of odd degree and has no root!!!
- $\tau_{\mathbb{F}_{p^k}}(n) = 2$ if $n!$ divides $k$: $\det A(x,1,0)$ has degree $n$ and has no root in $\mathbb{F}_{p^{n!}}$, which contains the splitting field of every polynomial of degree $n$!!!

# Bounds of $\tau_{\mathbb{F}}(n)$

Let $\rho_{\mathbb{F}}(n)$ be the maximum possible dimension of nonsingular vector subspace of $M_n(\mathbb{F})$. Then

$$\tau_{\mathbb{F}}(n) \leq \rho_{\mathbb{F}}(n) + 1 \quad (1)$$

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$A(x_1, \cdots, x_k)$ is $\mathbb{F}$-almost nonsingular $\Rightarrow A(0, x_2, \cdots, x_k)$ is $(k-1)$-dimensional nonsingular vector subspace of $M_n(\mathbb{F})$.

# The case of Real field

Let $\rho(n)$ be the Radon-Hurwitz number defined by

$$n = (2a+1)2^{b+4c} \Rightarrow \rho(n) = 2^b + 8c$$

For example,

$$\rho(32) = \rho(2^{1+4\cdot 1}) = 2^1 + 8 \cdot 1 = 10,$$
$$\rho(48) = \rho(16) = \rho(2^4) = 1 + 8 = 9$$

**Theorem (Adams, Lax and Phillips, 1963)**

$$\rho_{\mathbb{R}}(n) = \rho(n)$$

Hence

$$\tau_{\mathbb{R}}(n) \leq \rho(n) + 1$$

## Theorem

$$\tau_{\mathbb{R}}(n) = \rho(n) + 1$$

*if n has an odd divisor greater than 3.*
*Otherwise,*

$$\rho(2^{k-2}) + 1 \leq \tau_{\mathbb{R}}(2^k) \leq \rho(2^k) + 1,$$
$$\rho(2^{k-1}) + 1 \leq \tau_{\mathbb{R}}(3 \cdot 2^k) \leq \rho(2^k) + 1$$

Furthermore, if we denote $\tau_{\mathbb{R}}^{S}(n)$ to be the maximum possible number of indeterminates of a $\mathbb{R}$-almost nonsingular EPM which is **symmetric**. Then

$$\tau_{\mathbb{R}}^{S}(n) = \rho\left(\frac{n}{2}\right) + 1$$

provided $n$ has an odd divisor greater than 3.

THANK YOU!