Existence of nonsingular Entry pattern matrices

Hieu Ha Van
Supervised by: Dr. Rachel Quinlan
School of Maths, Statistics and Applied Maths

NUI Galway

March 22, 2018

Nonsingular entry pattern matrices

An entry pattern matrix (EPM for short) is a matrix in which:

- Each entry is an element of a specified set of independent indeterminates.
- Entries can be the same, but can not be a constant.

Nonsingular entry pattern matrices

An entry pattern matrix (EPM for short) is a matrix in which:

- Each entry is an element of a specified set of independent indeterminates.
- Entries can be the same, but can not be a constant.

A square EPM $A(x_1, \cdots, x_k)$ is said to be nonsingular over a field $\mathbb{F}$ (or $\mathbb{F}$-nonsingular) if $\det A(a_1, \cdots, a_k) \neq 0$ for all vector $(a_1, \cdots, a_k) \neq (constant)(1, \cdots, 1)$.

Singular EPMs	$\mathbb{F}_2$-nonsingular EPM	$\mathbb{R}$-nonsingular EPM
$\begin{bmatrix} x & x \\ x & y \end{bmatrix}, \begin{bmatrix} x & y \\ z & t \end{bmatrix}$	$\begin{bmatrix} x & y & y & y \\ y & x & y & y \\ x & x & x & y \\ x & x & y & x \end{bmatrix}$	$\begin{bmatrix} x & y & y & x & z & x \\ y & y & x & z & x & x \\ y & z & z & y & x & y \\ z & y & z & y & y & x \\ x & x & z & y & z & z \\ x & z & x & z & z & y \end{bmatrix}$

Nonsingular entry pattern matrices

An entry pattern matrix (EPM for short) is a matrix in which:

- Each entry is an element of a specified set of independent indeterminates.
- Entries can be the same, but can not be a constant.

A square EPM $A(x_1, \cdots, x_k)$ is said to be nonsingular over a field $\mathbb{F}$ (or $\mathbb{F}$-nonsingular) if $\det A(a_1, \cdots, a_k) \neq 0$ for all vector $(a_1, \cdots, a_k) \neq (constant)(1, \cdots, 1)$.

Singular EPMs	$\mathbb{F}_2$-nonsingular EPM	$\mathbb{R}$-nonsingular EPM
$\begin{bmatrix} x & x \\ x & y \end{bmatrix}, \begin{bmatrix} x & y \\ z & t \end{bmatrix}$	$\begin{bmatrix} x & y & y & y \\ y & x & y & y \\ x & x & x & y \\ x & x & y & x \end{bmatrix}$	$\begin{bmatrix} x & y & y & x & z & x \\ y & y & x & z & x & x \\ y & z & z & y & x & y \\ z & y & z & y & y & x \\ x & x & z & y & z & z \\ x & z & x & z & z & y \end{bmatrix}$

This talk is for the existence of nonsingular entry pattern matrices.

Equivalence conditions for nonsingular EPMs

Let $A(x_1, \cdots, x_k)$ be an $n \times n$ EPM. Then $A(x_1, \cdots, x_k) = A(x_1 - x_k, \cdots, x_{k-1} - x_k, 0) + x_k \mathfrak{J}$ where $\mathfrak{J} = \mathfrak{j}^T \mathfrak{j}$ is the matrix of ones (all-ones matrix) and $\mathfrak{j}$ is the row vector of ones. Hence, by matrix determinant lemma, we have

$$\begin{aligned}&\det A(x_1, \cdots, x_k)\\&= \det A(x_1 - x_k, \cdots, x_{k-1} - x_k, 0) \left(1 + x_k \cdot \mathfrak{j} \cdot A(x_1 - x_k, \cdots, x_{k-1} - x_k, 0)^{-1} \cdot \mathfrak{j}^T\right)\end{aligned}$$

Therefore,

Theorem

$A(x_1, \cdots, x_k)$ is $\mathbb{F}$-nonsingular if and only if $\det A(x_1, \cdots, x_{k-1}, 0) = 0 \Leftrightarrow x_1 = \cdots = x_{k-1} = 0$, *and* *mathfrakj* $\cdot A(x_1, \cdots, x_{k-1}, 0)^{-1} \cdot \mathfrak{j} = 0$ *for all* $(x_1, \cdots, x_{k-1}) \neq 0$.

Moreover, if $\det A(x_1, \cdots, x_{k-1}, 0) = 0$ then

$$\det \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \vdots & A \end{bmatrix}_{pn} \neq 0 \text{ and } \mathrm{j} \cdot \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \vdots & A \end{bmatrix}^{-1} \cdot \mathrm{j}^T \mod p = 0$$

This proves the following corollary.

Corollary

Let $\mathbb{F}$ be a field of positive characteristic p. If there exists a pattern matrix $A(x_1, \cdots, x_{k-1}, 0)$ of order n such that $\det A(x_1, \cdots, x_{k-1}, 0) = 0 \Rightarrow x_1 = \cdots = x_{k-1} = 0$, then there exists a nonsingular EPM of order pn with k indeterminates.

Constructions for nonsingular EPMs over $\mathbb{F}_p$

Let $\mathbb{F} = \mathbb{F}_p$ be the prime finite field of p elements. Then the Artin-Schreier polynomial

$$x^p - x - 1$$

is irreducible over $\mathbb{F}_p$.
Let α be a root of that polynomial. Then

$$\mathbb{F}_{p^p} = \langle 1, \alpha, \cdots, \alpha^{p-1} \rangle_{\mathbb{F}_p}$$

For any non-zero element $\beta \in \mathbb{F}_{p^p}$, we will define

$$f_\beta : \mathbb{F}_{p^p} \to \mathbb{F}_{p^p} \text{ given by } f_\beta(x) = \beta x$$

Then f_β is an automorphism for all non-zero β.

If we consider $\mathbb{F}_{p^p}$ as a p-dimensional vector space over $\mathbb{F}_p$ then the matrices of $f_1, f_{\alpha^2}, \cdots$ w.r.t. the basis $\{1, \alpha, \cdots, \alpha^{p-1}\}$ are

$$f_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{p\times p}, f_\alpha = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{p\times p}, \cdots$$

It is easily seen that $f_1 + f_\alpha + f_{\alpha^3} + \cdots + f_{\alpha^{p-2}}$ are $(0,1)$-matrix. Furthermore,

$$x_1 f_1 + x_2 f_{\alpha^3} + \cdots + x_{\frac{p+1}{2}} f_{\alpha^{p-2}} = f_{x_1 + x_2\alpha_3 + \cdots + x_{\frac{p+1}{2}}\alpha^{p-2}}$$

Hence, we may construct a $p^2 \times p^2$ $\mathbb{F}_p$-nonsingular EPM with $\frac{k+3}{2}$ indeterminates.

THANK YOU!