Postgraduate Modelling Research Group
School of Mathematics, Statistics and Applied Mathematics, NUI Galway

Numerical solution of a complex-valued singularly perturbed differential equation

Faiza Alssaedi
Supervisor: Niall Madden

February 22, 2018

Content

For more, see [Miller et al., 2012].

Introduction

We are interested in the numerical solution of a singularly perturbed, second-order, complex-valued reaction diffusion equation. Our model differential equation is

$$Lu := -\varepsilon^2 u'' + bu = f \qquad \text{on} \qquad \Omega = (0,1), \tag{1a}$$

subject to the boundary conditions

$$u(0) = 0, \qquad u(1) = 0. \tag{1b}$$

Here ε is a positive, real-valued parameter: $0 < \varepsilon \leq 1$, but typically $\varepsilon \ll 1$.

The functions b and f are complex valued functions on the real interval Ω. That is, $b : \Omega \to \mathbb{C}$, and $f : \Omega \to \mathbb{C}$.

Why this is interesting

These problems are interesting, and difficult, becuase solutions feature boundary layers.

There are many methods for solving problems such as (1). But it is always assumed that the coefficients and solution are real-valued.

Furthermore, it is usually assumed that $b(x) > 0$ for all x. A consequence of this is that differential operator satisfies a maximum principle.

This means that, for example, if $f(x) > 0$, then $u(x) > 0$. (I have spoken about this at previous talks).

But in the complex-valued case, we don't have an analogous concept.

An example with layers

Example (Layers)

$$-\varepsilon^2 u'' + (i+4)^2 u = (4+4i)e^x \quad \text{on } (0,1), \qquad u(0) = u(1) = 0. \qquad (2)$$

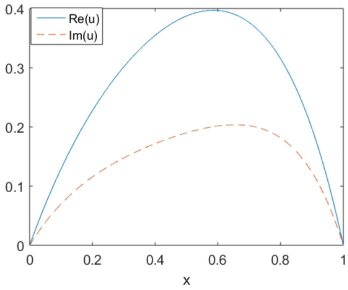


Figure: Real and imaginary parts of the solutions to (2) with $\varepsilon = 1$.

An example with layers

Example (Layers)

$$-\varepsilon^2 u'' + (i+4)^2 u = (4+4i)e^x \quad \text{on } (0,1), \qquad u(0) = u(1) = 0. \qquad (2)$$

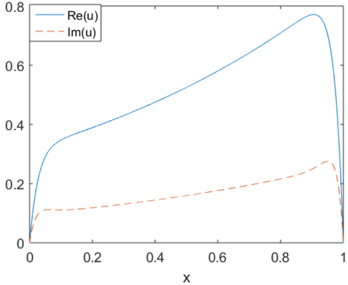


Figure: Real and imaginary parts of the solutions to (2) with $\varepsilon = 0.1$.

But the solution has oscillations

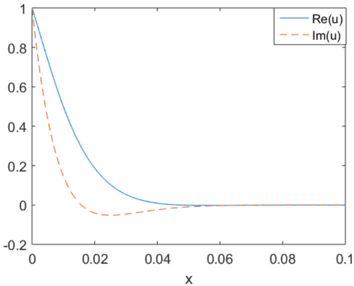
Example (Oscillations)

$$-\varepsilon^2 u'' + (1 + \frac{i}{2})^2 u = 0 \quad u(0) = 1 + i, u(1) = 0. \tag{3}$$

The solution is (roughly)

$e^{-x/\varepsilon}(\cos(x/2\varepsilon)+\sin(x/2\varepsilon))+ie^{-x/\varepsilon}(\cos(x/2\varepsilon)-\sin(x/2\varepsilon))+\mathcal{O}(e^{-1/\varepsilon})$.

The imaginary part of u is neither positive nor monotonic.



Since we can’t use standard ideas, we will take a special approach:

- Rewrite the equation as a coupled system of real-valued problems;
- Follow some analysis from [Kellogg et al., 2008]
- That in turn uses arguments from classic paper [Bahvalov, 1969].

A system of reaction-diffusion equations

We now consider the following system by rewriting (1) as

$$-\varepsilon^2(u_r + iu_i)'' + (b_r + ib_i)(u_r + iu_i) = f_r + if_i, \quad (4)$$

From (4), when we equate real terms and imaginary terms separately, we get

$$-\varepsilon^2 u_r'' + b_r u_r - b_i u_i = f_r,$$
$$-\varepsilon^2 u_i'' + b_i u_r + b_r u_i = f_i.$$

So, we can write the system as

$$\vec{L}\vec{u} := -\varepsilon^2 \vec{u}'' + B\vec{u} = \vec{f}, \quad (5)$$

We can write the system as

$$\vec{L}\vec{u} := -\varepsilon^2 \vec{u}'' + B\vec{u} = \vec{f}, \tag{6}$$

where

$$\vec{u} = \begin{pmatrix} u_r \\ u_i \end{pmatrix}, \; B = \begin{pmatrix} b_r & -b_i \\ b_i & b_r \end{pmatrix} \quad \text{and } \vec{f} = \begin{pmatrix} f_r \\ f_i \end{pmatrix}.$$

Since $u = u_r + iu_i$, the bounds on the solution and its derivatives are given by

$$\|\vec{u}^{(k)}\|_{\bar{\Omega}} \leq C(1 + \varepsilon^{-k}),$$

where $0 \leq k \leq 4$.

Lemma

Assume that $b_r > 0$. Then the matrix B is coercive, meaning that, there exists a constant α such that $\sqrt{b_r} \geq \alpha > 0$ and

$$\vec{v}^T B \vec{v} \geq \alpha^2 \vec{v}^T \vec{v} \quad \text{for all } \vec{v} \in \mathbb{R}^2. \tag{7}$$

proof

The finite difference method for equation (6) is: find $\vec{U}$ such that

$$\vec{L}^N \vec{U}_j := -\varepsilon^2 \delta^2 \vec{U}_j + B\vec{U}_j = \vec{f}_j \quad \text{for} \quad j = 1, ..., N-1, \qquad (8)$$
$$\vec{U}_0 = \vec{u}(0), \quad \vec{U}_N = \vec{u}(1),$$

where $\vec{U}_j$ is the approximation for $\vec{u}(x_j)$ and a standard second-order approximation of the second derivative is

$$\delta^2 \vec{U}_j = \frac{1}{\hbar_j}\left(\frac{\vec{U}_{j-1}}{h_j} - \vec{U}_i\left(\frac{1}{h_j} + \frac{1}{h_{j+1}}\right) + \frac{\vec{U}_{j+1}}{h_{j+1}}\right), \qquad (9)$$

where $\hbar_j = (x_{j+1} - x_{j-1})/2$.

Lemma

The discrete operator $\vec{L}^N$ satisfies the stability inequality

$$\|\vec{W}\|_{\bar{\Omega}} \leq \alpha^{-2}\|\vec{L}^N\vec{W}\|_{\Omega} + \|\vec{W}(0)\| + \|\vec{W}(1)\|,$$

for arbitrary vector-valued functions $\vec{W}$ defined on $\bar{\Omega}$.

Theorem

Let Ω^N be the Bakhvalov mesh, and let $\vec{U}$ be the solution to (8) *on this mesh. Then, if $\vec{u}$ solves* (6)*,*

$$\|\vec{u} - \vec{U}\| \leq CN^{-2}. \tag{10}$$

Numerical results

We now present the results for problem (2):

$$-\varepsilon^2 u'' + (4+i)^2 u = (4+4i)e^x \qquad u(0) = 0, \qquad u(1) = 0.$$

Table: Errors, E_ε^N, for problem (2), solved on a Shishkin mesh.

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
1	1.105e-03	2.781e-04	6.961e-05	1.741e-05	4.353e-06
1e-01	3.988e-02	2.050e-02	5.595e-03	1.468e-03	3.694e-04
1e-02	2.430e-02	3.863e-02	3.656e-02	1.954e-02	6.615e-03

Table: Errors, E_ε^N, for problem (2), solved on a Bakhvalov mesh.

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
1	1.105e-03	2.781e-04	6.961e-05	1.741e-05	4.353e-06
1e-01	1.258e-02	3.321e-03	8.419e-04	2.112e-04	5.291e-05
1e-02	1.257e-02	3.319e-03	8.415e-04	2.111e-04	5.288e-05

References

[Bahvalov, 1969] Bahvalov, N. S. (1969).
On the optimization of the methods for solving boundary value problems in the presence of a boundary layer.
Ž. Vyčisl. Mat. i Mat. Fiz., 9:841–859.

[Kellogg et al., 2008] Kellogg, R. B., Linss, T., and Stynes, M. (2008).
A finite difference method on layer-adapted meshes for an elliptic reaction-diffusion system in two dimensions.
Math. Comp., 77(264):2085–2096.

[Miller et al., 2012] Miller, J. J. H., O'Riordan, E., and Shishkin, G. I. (2012).
Fitted numerical methods for singular perturbation problems.
World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, revised edition.

Thank you