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Parameter robust methods for second-order complex-valued reaction-diffusion equations

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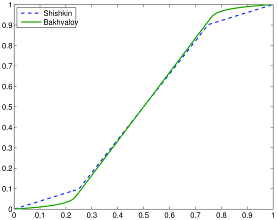


§Outline

For more, see [2].

We are interested in the numerical solution of a singularly perturbed, second-order, complex-valued reaction diffusion equation. Our model differential equation is

$$Lu := -\varepsilon^2 u'' + bu = f \qquad \text{on} \qquad \Omega = (0, 1), \tag{1a}$$

subject to the boundary conditions

$$u(0) = 0, \qquad u(1) = 0. \tag{1b}$$

Here ε is a positive, real-valued parameter: $0 < \varepsilon \leqslant 1$, but typically $\varepsilon \ll 1$.

The functions b and f are complex valued functions on the real interval Ω. That is, $b : \Omega \to \mathbb{C}$, and $f : \Omega \to \mathbb{C}$. More precise assumptions on b and f are made below.

A motivating example

We consider the following example: find $u \in \mathbb{C}^2(\Omega)$ such that

$$-\varepsilon^2 u'' + (1+4i)^2 u = (4+4i)e^x \quad \text{on } \Omega = (0,1), \quad u(0) = u(1) = 0. \tag{2}$$

The exact solution can be expressed as

$$u(x) = C_1 e^{\frac{-(4+i)x}{\varepsilon}} + C_2 e^{\frac{(4+i)(x-1)}{\varepsilon}} + \frac{(4+4i)e^x}{-\varepsilon^2 + 15 + 8i}, \tag{3}$$

where

$$C_1 = -\frac{4(i - ie^{\frac{(\varepsilon-4-i)}{\varepsilon}} + 1 - e^{\frac{(\varepsilon-4-i)}{\varepsilon}})}{(-\varepsilon^2 + 15 + 8i)(1 - e^{\frac{(-8-2i)}{\varepsilon}})} \text{ and } C_2 = \frac{4(ie^{\frac{(-4-i)}{\varepsilon}} - ie + e^{\frac{(-4-i)}{\varepsilon}} - e)}{(-\varepsilon^2 + 15 + 8i)(1 - e^{\frac{(-8-2i)}{\varepsilon}})}.$$

The first two terms on the right-hand side of (3) correspond to the left and right layer, respectively.

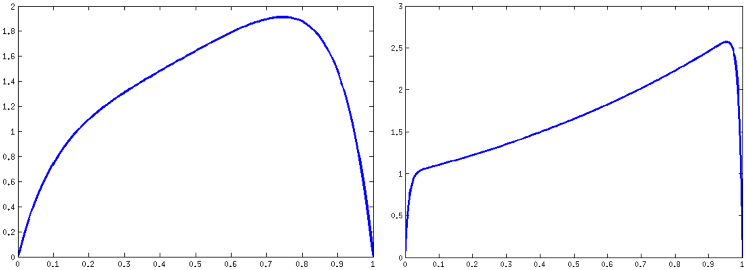


Figure: Real and imaginary parts of the solutions to (2) with $\varepsilon = 1$ (left) and $\varepsilon = 0.1$ (right)

In Figure 1 we show u with $\varepsilon = 1$ (left), which does not features layers. On the right for smaller ε (in this case $\varepsilon = 0.1$), the solutions possess boundary layers near $x = 0$ and $x = 1$, in both the real and the imaginary parts.

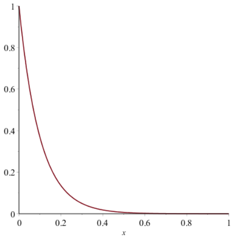
Definition

The differential operator L satisfies a **maximum principle**, if $\psi(0) \geqslant 0$ and $\psi(1) \geqslant 0$, and $L\psi(x) \geqslant 0$, for all $x \in \Omega$, imply that $\psi(x) \geqslant 0$, for all $x \in \bar{\Omega}$ [2].

To see how a problem such as (1) differs from the real-valued case, consider the example

$$-\varepsilon^2 u'' + u = 0 \qquad u(0) = 0, \qquad u(1) = 1. \tag{4}$$

The solution is $u(x) \cong e^{-x/\varepsilon}$, which is positive and monotonic. The associated differential operator satisfies a maximum principle.



Comparing Real-Valued and Complex-Valued Problems

Now consider the complex-valued problem:

$$-\varepsilon^2 u'' + (1+i)^2 u = 0 \qquad u(0) = 1+i, \qquad u(1) = 0. \tag{5}$$

The solution is $u(x) \cong e^{-x/\varepsilon}(\cos(-x/\varepsilon) + i\cos(-x/\varepsilon))$,thus, neither the real and the imaginary part of u are neither positive nor monotonic. The associated differential operator does not satisfy a maximum principle, in the conventional sense, and the solution can oscillate.

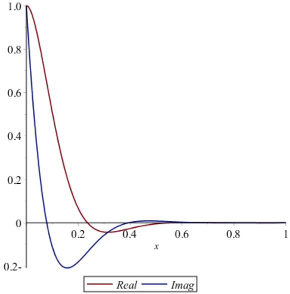


Figure: The solutions to (5) with $\varepsilon = 0.1$.

Lemma

Let u *be the solution of* (1)*. Then, for* $0 \leqslant k \leqslant 4$,

$$\|u^{(k)}\| \leqslant C(1 + \varepsilon^{-k}), \tag{6}$$

Proof

On a arbitrary mesh, $\Omega^N := \{0 = x_0 < x_1 < \cdots < x_N = 1\}$, where $h_i = x_i - x_{i-1}$. suppose we want to approximate u_i' by a finite difference approximation based only on values of u at finite number of points near x_i. One obvious choice would be to use the forward difference approximation:

$$D^+U_i = \frac{U_{i+1} - U_i}{h_{i+1}}$$

Note that D^+U_i is the slope of the line interpolating u at points x_i and x_{i+1}. Another one-sided approximation would be the backward difference approximation:

$$D^-U_i = \frac{U_i - U_{i-1}}{h_i}$$

Finally, we have the centred approximation

$$D^0U_i = \frac{1}{2}(D^+U_i + D^-U_i) = \frac{1}{2}\left(\frac{U_{i+1}}{h_{i+1}} + U_i(\frac{1}{h_i} - \frac{1}{h_{i+1}}) - \frac{U_{i-1}}{h_i}\right)$$

This is the slope of the line interpolating u at x_{i+1} and x_{i-1}.

A standard second-order approximation of a second derivative is

$$\delta^2 U_i := \frac{1}{\hbar_i}\left(\frac{U_{i-1}}{h_i} - U_i(\frac{1}{h_i} + \frac{1}{h_{i+1}}) + \frac{U_{i+1}}{h_{i+1}}\right),$$

where $h_i = x_i - x_{i-1}$ and $\hbar_i = (x_{i+1} - x_{i-1})/2$. The finite difference operator is defined as:

$$L^N \psi_i := -\varepsilon \delta^2 \psi_i + b(x_i)\psi_i \quad \text{for} \quad i = 1, \dots, N-1.$$

The finite difference method is:

$$\begin{aligned} U_i(0) &= u_0, \\ -\varepsilon^2 \delta^2 U_i + b(x_i)U_i &= f(x_i), \quad \text{for all} \quad x_i \in \Omega^N, \\ U_i(1) &= u_1. \end{aligned} \tag{7}$$

We construct a standard *Shishkin* mesh with the mesh parameter $\tau = \min\{\frac{1}{4}, 2\frac{\varepsilon}{\beta}\ln N\}$, where $0 < \beta^2 \leqslant \min_{0\leqslant x\leqslant 1} b(x)$. We now define two mesh transition points at $x = \tau$ and $x = 1-\tau$. That is, we form a piecewise uniform mesh with $N/4$ equally-sized mesh intervals on each of $[0,\tau]$ and $[1-\tau,1]$, and $N/2$ equally-sized mesh intervals on $[\tau, 1-\tau]$. Typically, when ε is small, $\tau \ll 1/4$, the mesh is very find near the boundaries, and coarse in the interior.

The mesh may also be specified in terms of a mesh generating function, which we now define.

Definition

[1, p5] A strictly monotone function $\varphi : [0,1] \to [0,1]$ that maps a uniform mesh $t_i = i/N, i = 0, ..., N$, onto a layer-adapted mesh by $x_i = \varphi(t_i), i = 0, ..., N$, is called a **mesh generating function**.

The mesh generating function φ, for Shishkin mesh described above, is

$$\varphi(t) = \begin{cases} 4t\tau & t \leqslant \frac{1}{4}, \\ 2(1-\tau)(t-\frac{1}{4}) + 2\tau(\frac{3}{4}-t) & \frac{1}{4} < t < \frac{3}{4}, \\ 4(1-\tau)(1-t) + 4(t-\frac{3}{4}) & t \geqslant \frac{3}{4}. \end{cases}$$

Notice that this is a piecewise linear function.

We construct a standard *Bakhvalov* mesh with mesh parameters at $\sigma > 0$, $q \in (0, 1/2)$, typical values of the mesh parameters are $\sigma = 2$, $q = 1/4$, where the method has order σ and q is the proportion of mesh points in the layer. Mesh points are $x_i = i/N$ if $\sigma\varepsilon \geqslant \beta q$. However when $\sigma\varepsilon < \beta q$ one sets

$$x_i = \begin{cases} \varphi(i/N) & \text{for } i \leqslant N/2, \\ 1 - \varphi(N-i)/N & \text{for } i > N/2, \end{cases}$$

with a mesh generating function φ defined by

$$\varphi(t) = \begin{cases} \chi(t) := -\frac{2\varepsilon}{\beta} \ln(1 - \frac{t}{q}) & \text{for } t \in [0, \tau], \\ \pi(t) := \chi(\tau) + \chi'(\tau)(t - \tau) & \text{for } t \in [\tau, 1/2]. \end{cases}$$

where the point τ satisfies

$$\chi'(\tau) = \frac{1 - 2\chi(\tau)}{1 - 2\tau}.$$

This defines the mesh on $[0, 1/2]$ and it is extended to $[0, 1]$ by reflection about $x = 1/2$.

Numerical results

We now present the results for problem (2):

$$-\varepsilon^2 u'' + (4+i)^2 u = (4+4i)e^x \qquad u(0) = 0, \qquad u(1) = 0.$$

Table: Errors, E_ε^N, for problem (2), solved on a Shishkin mesh.

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
1	1.105e-03	2.781e-04	6.961e-05	1.741e-05	4.353e-06
1e-01	3.988e-02	2.050e-02	5.595e-03	1.468e-03	3.694e-04
1e-02	2.430e-02	3.863e-02	3.656e-02	1.954e-02	6.615e-03
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Table: Errors, E_ε^N, for problem (2), solved on a Bakhvalov mesh.

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1	1.105e-03	2.781e-04	6.961e-05	1.741e-05	4.353e-06
1e-01	1.258e-02	3.321e-03	8.419e-04	2.112e-04	5.291e-05
1e-02	1.257e-02	3.319e-03	8.415e-04	2.111e-04	5.288e-05
1e-03	1.257e-02	3.319e-03	8.415e-04	2.111e-04	5.288e-05

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