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A short note on 4th order real-valued singularly perturbed problems

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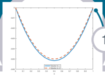
Content



Introduction

We are interested in the numerical solution of a singularly perturbed, fourth-order ordinary differential equations.

Our model differential equation is

$$-\varepsilon^2 u^{(4)}(x) + au''(x) - bu(x) = f(x) \qquad \text{on} \qquad \Omega := (0,1), \quad (1)$$

subject to the boundary conditions

$$u(0) = u''(0) = 0, \qquad u(1) = u''(1) = 0.$$

Here ε is a positive, real-valued parameter: $0 < \varepsilon \leq 1$, but typically $\varepsilon \ll 1$. And so the problem is **singularly perturbed**.
The coefficient functions *a*, *b* and right-hand side function *f* are real or complex-valued functions on the interval Ω.

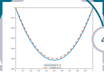
Research Question

The above problem is complicated, because solutions feature boundary layers.

The numerical methods are used to solve this problem (1)

- By using standard finite difference methods, on specialised fitted meshes: the well-known piecewise uniform *Shishkin* mesh, and the more complicated *Bakhvalov* mesh.
- The numerical analysis of such method usually relies on Maximum Principles, but these do not hold, in a direct way.
- Since we cannot use standard ideas, we take the approach of rewriting (1) as a coupled system of real-valued problems, and establish that the coefficient matrix for this system is positive definite.

The finite difference method



Consider an arbitrary mesh, $\Omega^N := \{0 = x_0 < x_1 < \cdots < x_N = 1\}$. On this we define the standard second-order approximation of the second derivative

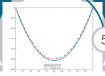
$$\delta^2 u_i := \frac{1}{\hbar_i}\left(\frac{u_{i-1}}{h_i} - u_i\left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right) + \frac{u_{i+1}}{h_{i+1}}\right), \qquad (2)$$

and we define the standard 4th-order approximation of the fourth derivative

$$u^{(4)}(x_i) \approx D^4 u_i := \frac{6}{(\hbar_{i-1} + \hbar_{i+1})(h_{i-1} + h_i + h_{i+1})\hbar_{i-1} h_{i-1}} u_{i-2}$$
$$- \frac{12}{h_{i-1}(h_i + h_{i+1} + h_{i+2})\hbar_i h_i} u_{i-1} + \frac{6}{\hbar_{i-1} h_2 \hbar_{i+1} h_3} u_i - \frac{12}{\hbar_i h_{i+1} h_{i+2}(h_{i-1} + h_i + h_{i+}}$$
$$+ \frac{6}{(\hbar_{i-1} + \hbar_{i+1})(h_i + h_{i+1} + h_{i+2})\hbar_{i+1} h_{i+2}} u_{i+2}, \qquad (3)$$

where $h_i = x_i - x_{i-1}$ and $\hbar_i = (x_{i+1} - x_{i-1})/2$.

Real-valued problems



We consider the case of real-valued fourth-order ordinary differential equations and the following example from the literature.

[Shanthi and Ramanujam, 2002]:

$$-\varepsilon^2 u^{(4)}(x) + au''(x) - bu(x) = f(x) \qquad \text{on} \qquad \Omega := (0,1), \quad (4)$$

subject to the boundary conditions

$$u(0) = u''(0) = u(1) = u''(1) = 0.$$

The coefficient functions a, b and right-hand side function f are real-valued functions on the interval Ω.

About the boundary conditions

It is important to note that the boundary conditions are

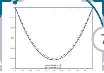
$$u(0) = u''(0) = u(1) = u''(1) = 0. \tag{5}$$

It is also common to have boundary conditions of the form

$$u(0) = u'(0) = u(1) = u'(1) = 0. \tag{6}$$

When the conditions are of the form given in (5), then the equation can be written as a system of two second order equations, and solved using techniques for such problems.

How *not* to solve this problem



When the boundary conditions are

$$u(0) = u''(0) = u(1) = u''(1) = 0.$$

we can set $u'' = w$ and write the problem as the following coupled system:

$$-\varepsilon^2 w'' + aw - bu = f, \tag{7a}$$

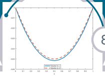
$$-u'' + w = 0, \tag{7b}$$

Written in matrix-vector form, this is

$$-\begin{pmatrix} \varepsilon^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w'' \\ u'' \end{pmatrix} + \begin{pmatrix} a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

There are numerous papers that study this formulation, but some are flawed. For example, in [Xenophontos et al., 2013a] some of the analysis depends on the coefficient matrix of the zero-order term being "pointwise positive definite (but not necessarily symmetric)". But this is impossible!

A positive definite system



We now propose an simple approach that allows one to reformulate (4) as a system with a (non-symmetric) positive definite coefficient matrix. We set

$$u'' := \alpha w + \beta u,$$

and, consequently,

$$u^{(4)} = \alpha w'' + \beta \alpha w + \beta^2 u.$$

With this, (4) can be transformed as a system of two equations of the form

$$-\varepsilon^2 \alpha w'' + (a\alpha - \varepsilon^2 \alpha \beta) w + (a\beta - \varepsilon^2 \beta^2 - b) u = f, \tag{8a}$$

$$-u'' + \alpha w + \beta u = 0, \tag{8b}$$

subject to the boundary conditions

$$u(0) = w(0) = u(1) = w(1) = 0.$$

Coercivity property of the matrix

Written in matrix form, this is

$$-\begin{pmatrix} \varepsilon^2\alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w'' \\ u'' \end{pmatrix} + B \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

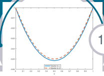
where

$$B = \begin{pmatrix} a\alpha - \varepsilon^2\alpha\beta & a\beta - \varepsilon^2\beta^2 - b \\ \alpha & \beta \end{pmatrix}.$$

Our eventual goal is to analyse the convergence of a finite difference scheme for this problem. We wish to apply the analysis techniques from [Bakhvalov, 1969, Kellogg et al., 2008], for which we need that

$$\mathbf{v}^T B\mathbf{v} \geq \delta \mathbf{v}^T \mathbf{v},$$

for all vectors **v**, and some positive constant δ.



This B satisfies $\mathbf{v}^T B\mathbf{v} \geq \delta\mathbf{v}^T\mathbf{v}$, for all v, if and only if, $M = (B^T + B)/2$ is symmetric positive definite. Here

$$M = \begin{pmatrix} a\alpha - \varepsilon^2\alpha\beta & \frac{1}{2}(a\beta - \varepsilon^2\beta^2 - b + \alpha) \\ \frac{1}{2}(a\beta - \varepsilon^2\beta^2 - b + \alpha) & \beta \end{pmatrix}.$$

Clearly, M is symmetric. In addition M is positive definite if and only if all of its eigenvalues are positive [Horn et al., 1990, Thm 7.2.1]. That will be the case if M is *strictly diagonally dominant*, with positive diagonal entries, i.e., $M_{ii} > \sum_{j\neq i} |M_{ij}|$ for $i = 1, 2$ [Beezer, 2008]. So, thus, we require that

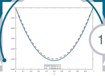
(i) $|a\alpha - \varepsilon^2\alpha\beta| > 0$,

(ii) $|\beta| > 0$,

(iii) $|a\alpha - \varepsilon^2\alpha\beta| > |\frac{1}{2}(a\beta - \varepsilon^2\beta^2 - b + \alpha)|$,

(iv) $|\beta| > |\frac{1}{2}(a\beta - \varepsilon^2\beta^2 - b + \alpha)|$.

Example



Now consider the real-valued problem:

$$-\varepsilon^2 u^{(4)}(x) + 2u''(x) - 4u(x) = f(x) \qquad \text{on} \qquad \Omega := (0,1), \quad (9)$$

subject to the boundary conditions

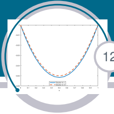
$$u(0) = u''(0) = 0, \qquad u(1) = u''(1) = 0.$$

If we take $\alpha = 1$ and $\beta = 1$, we have

$$B = \begin{pmatrix} 2-\varepsilon^2 & -(\varepsilon^2+2) \\ 1 & 1 \end{pmatrix}, \ \textit{and} \quad M = \begin{pmatrix} 2-\varepsilon^2 & -\frac{1}{2}(\varepsilon^2+1) \\ -\frac{1}{2}(\varepsilon^2+1) & 1 \end{pmatrix}.$$

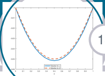
where M is a symmetric positive definite for $\varepsilon < \sqrt{2}$.

Conclusions and future work



- We have shown that standard ideas cannot be used for this problem.
- We are now working on the analysis of methods for fourth-order complex-valued problems in the case where the problem can be re-cast as a coupled system of second-order problems (see, e.g., [Xenophontos et al., 2013b]).
- We also aim to extend the work to complex-valued fourth-order ones which cannot be written as a coupled system of second-order ones (e.g., [Constantinou et al., 2016]).

References



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Thank you