Error Distributions in the Square Root Map with Additive Noise

Eoghan Staunton, Petri T. Piiroinen

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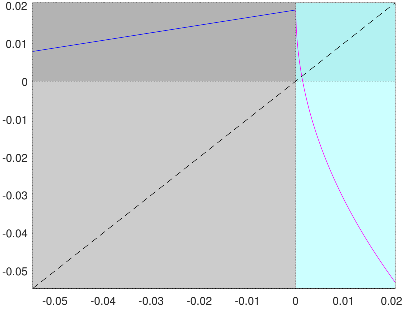
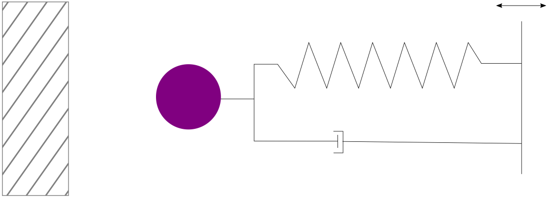
The Square Root Map

Many impacting systems, including impact oscillators, are described by a 1-D map known as the square root map.

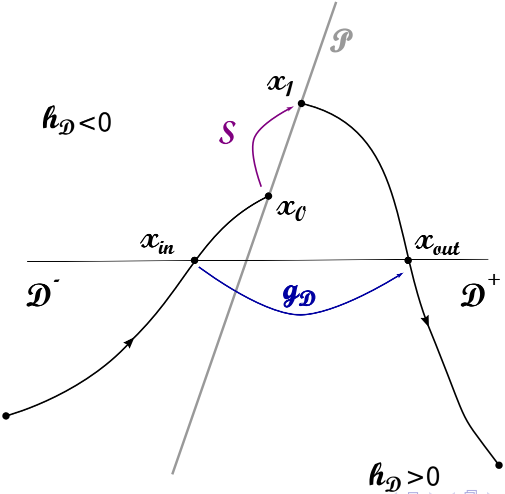
$$x_{n+1} = S(x_n) = \begin{cases} \mu + b x_n & \text{if } x_n < 0 \\ \mu - a\sqrt{x_n} & \text{if } x_n \geq 0 \end{cases}$$





The Square Root Map

This continuous, nonsmooth map can be derived as an approximation for solutions of piecewise smooth differential equations near certain types of grazing bifurcation.

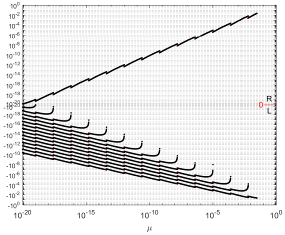


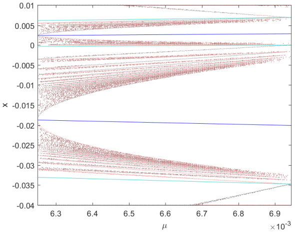
Multistability In the Square Root Map

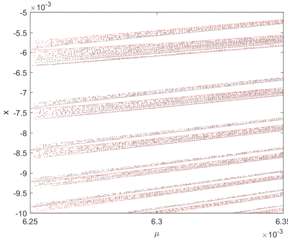
If $0 < b < \frac{1}{4}$ there are values of $\mu > 0$ for which

- a single stable periodic orbit of period m, with code $(RL^{m-1})^\infty$, exists for each $m = 2, 3, \ldots$
- two stable periodic orbits, one of period m, with code $(RL^{m-1})^\infty$, and the other of period $m+1$, with code $(RL^m)^\infty$, exist for each $m = 2, 3, \ldots$

These are the only possible attractors of the system except at bifurcation points.







Types of Noise

In two separate papers Simpson, Hogan and Kuske and Simpson and Kuske make a careful analysis of how noise in impacting systems manifests in the map. They conclude that there are several different models. We focus on the simplest model of Gaussian white noise with noise amplitude Δ.

$$x_{n+1} = S_a(x_n) = \begin{cases} \mu + bx_n + \xi_n & \text{if } x_n < 0 \\ \mu - a\sqrt{x_n} + \xi_n & \text{if } x_n \geq 0 \end{cases} \tag{1}$$

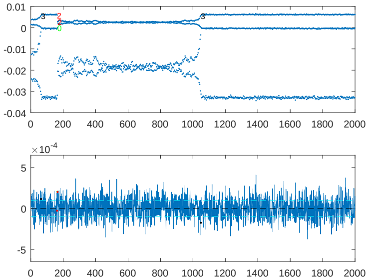
where $\xi_n \sim N(0, \Delta^2)$.

The Transition Mechanism

Perhaps the most interesting phenomenon that we have observed in noisy systems of this form is the potential for repeated intervals of persistent RL^{m-1} dynamics in a noisy system with $\mu < \mu_m^s$.
In particular we observe transitions of the following form for μ in a neighbourhood of μ_m^s such that $\mu < \mu_m^s$.

$$RL^m RL^m \dots \underline{RL^m RL^{m-1} RL^{k-2} RL^{m-1} RL^{m-1}} \dots RL^{m-1} \qquad (2)$$

for $k \in \{2, 3, \dots, m\}$.



The Error Distributions

Let

$$z_0 = x_0 = R_{m+1},\ z_{n+1} = S(z_n),\ x_{n+1} = S_\alpha(x_n) \text{ and } \epsilon_n = x_n - z_n. \quad (3)$$

In order for such a transition to occur we require $x_{n(m+1)-1} > 0$ for some $n \in \mathbb{N}$. Since $z_{n(m+1)-1} = L^m_{m+1}$ for all n, this means that we require $\epsilon_{n(m+1)-1} > -L^m_{m+1}$ for some $n \in \mathbb{N}$ such that $\epsilon_{n_0(m+1)-1} < -L^m_{m+1}$ for all $n_0 < n$. Iteratively we can find the following expressions for such errors:

$$\begin{aligned}
\epsilon_m &= \sum_{i=0}^{m-1} b^{m-1-i}\xi_i, \text{ and} \\
\epsilon_{(n+1)(m+1)-1} &= ab^{m-1}\left(\sqrt{R_{m+1}} - \sqrt{R_{m+1} + b\epsilon_{n(m+1)-1} + \xi_{n(m+1)-1}}\right) \\
&\quad + \sum_{i=n(m+1)}^{(n+1)(m+1)-2} b^{(n+1)(m+1)-2-i}\xi_i, \text{ for } n \geq 1. \qquad (4)
\end{aligned}$$

The Error Distributions

The probability of such a transition being induced by an appropriate error within $N(m+1)-1$ iterates is given by the sum

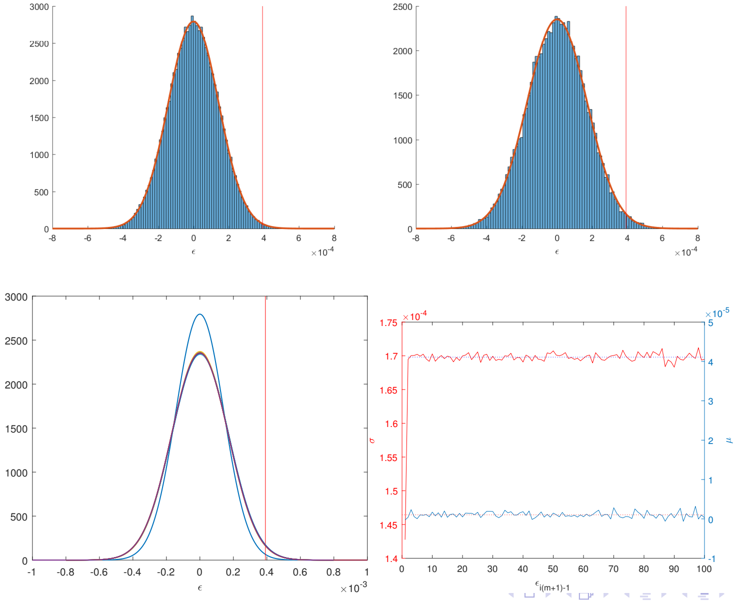
$$P_{\sum_N} = \sum_{n=1}^{N-1} P\left(\epsilon_{(n+1)(m+1)-1} > -L_{m+1}^{m} \middle| \cap_{k=1}^{n} \left(\epsilon_{k(m+1)-1} < -L_{m+1}^{m}\right)\right). \quad (5)$$

We are currently considering the individual probabilities

$$P_{n+1} = P\left(\epsilon_{(n+1)(m+1)-1} > -L_{m+1}^{m} \middle| \cap_{k=1}^{n} \left(\epsilon_{k(m+1)-1} < -L_{m+1}^{m}\right)\right) \quad (6)$$

and trying to construct their distributions.

The Error Distributions



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