Computation Steenrod Square of finite groups

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Overveiw

Tensor product

Let M and N be two $\mathbb{Z}$-module over arbitrary ring . Their *tensor product* $M \otimes_{\mathbb{Z}} N$ is $\mathbb{Z}$-module generated by formal symbols $m \otimes n$ for some $m \in M$, $n \in N$ subject to the relations.

- $(m_1 + m_2) \otimes n = (m_1 \otimes n) + (m_2 \otimes n)$,
- $m \otimes (n_1 + n_2) = (m \otimes n_1) + (m \otimes n_2)$
- $r.m \otimes n = m \otimes r.n$, for all $r \in \mathbb{Z}$, $m_1, m_2 \in M$ and $n_1, n_2 \in N$.

Tensor product of two chain complex

[1, 3] Suppose that (C_*, ∂) and (D_*, ∂) are two chain complexs of free R-module, let

$$(C_* \otimes_R D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

where the tensor product $C_p \otimes_R D_q$ is the module of Definition3. one can check that $(C_* \otimes_R D_*)_n$ is the free R-module with basis $a^p_\lambda \otimes b^q_\mu$ where $a^p_\lambda \in C_p$ and $b^q_\mu \in D_q$, $p, q \geq 0,\ p + q = n$. The boundary oprator

$$d_n : (C_* \otimes_R D_*)_n \longrightarrow (C_* \otimes_R D_*)_{n-1}$$

be the homomorphism define on free generators by

$$d_n(a^p_\lambda \otimes b^q_\mu) = \partial a^p_\lambda \otimes b^q_\mu + (-1) a^p_\lambda \otimes \partial b^q_\mu.$$

tensor product of resolution

Let A be a $\mathbb{Z}G$-module and let B be a $\mathbb{Z}H$-module for two groups G, H. By regarding both A and B as abelian groups, we can form their tensor product $A \otimes_{\mathbb{Z}} B$. There is an action of the direct product of groups $G \times H$ on the tensor product $A \otimes_{\mathbb{Z}} B$ define by

$$(g,h)\cdot(a\otimes b) = (ga)\otimes(g'b), (g,h)\in G\times H, (a\otimes b)\in A\otimes_{\mathbb{Z}} B. \tag{1}$$

If A is a free $\mathbb{Z}G$-module and B is a free $\mathbb{Z}H$-module then $A \otimes_{\mathbb{Z}} B$ is a free $\mathbb{Z}(G \times H)$-module under this action. Let R_*^G be a chain complex of $\mathbb{Z}G$-modules and let S_*^H be a chain complex of $\mathbb{Z}H$-modules. By regarding these chain complexes as complexes of abelian groups we can form their tensor product $R_*^G \otimes_{\mathbb{Z}} S_*^H$ as defined in 3. Under the action 1, the chain complex $R_*^G \otimes_{\mathbb{Z}} S_*^H$ is a chain complex of $\mathbb{Z}(G \times H)$-modules.

cohomology operation

cohomology operation

A cohomology operation of type (G, n, G', m) is a natural transformation, $\phi : H^n(-, G) \longrightarrow H^m(-, G')$, that to any spaces, A, B and to any map $f : A \longrightarrow B$ there are functions ϕ_A, ϕ_B satisfying the naturality condition $f^*\phi_B = \phi_A f^*$.
The cohomology of a space B with coefficients in the field of p elements admits certain homomorphisms of operation β of type $(\mathbb{Z}_p, n, \mathbb{Z}_p, n+1)$, known as the Bockstein homomorphism

$$\beta : H^n(B, \mathbb{Z}_p) \to H^{n+1}(B, \mathbb{Z}_p),$$

the Steenrod squares are cohomology operations of type $(\mathbb{Z}_p, n, \mathbb{Z}_p, n+i)$

$$Sq^i : H^n(B, \mathbb{Z}_2) \to H^{n+i}(B, \mathbb{Z}_2), \quad i \geq 0.$$

for $p = 2$.

Cup product

The Eilenberg-Zilber theorem

The chain maps $\phi : C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$and $\times : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y)$ are natural homotopy equivalances which are naturally homotopy inverses of one another.

Definition

Let $\triangle : X \longrightarrow X \times X$, $x \longrightarrow (x, x)$ be the diagonal map. Then the cup product is the homomorphism

$$\cup : H^p(X) \otimes H^q(X) \longrightarrow H^{p+q}$$

The chain complex $\triangle_* : C_*(X) \longrightarrow C_*(X) \otimes C_*(X)$, and the cochain complex $\triangle^* : C^*(X) \otimes C^*(X) \longrightarrow C^*(X)$. One can define the cup product on the cochain by

$$(f \cup g)(c) = f \otimes g(\phi(\triangle_*(c)))$$

where $f \in C^p(X, \mathbb{Z})$ and $g \in C^q(X, \mathbb{Z})$, $c \in C_{p+q}(X)$ and ϕ is the Eilenberg-Zilber map.

Cup$i$-product

Construct cup-i product

The universal covering space $\tilde{B} = E_G$ is contractible, and that the chain complex $C_*(\tilde{B})$ is thus exact. Let $C_2 = \langle t : t^2 = 1 \rangle$ is the group of order 2 generated by t. Let $B = B_G$ for some group G and set $R_*^G = C_*(\tilde{B})$. The group C_2 acts on $R_*^G \otimes_{\mathbb{Z}} R_*^G$ by the interchange map

$$\tau : R_*^G \otimes R_*^G \longrightarrow R_*^G \otimes R_*^G,$$

$$t \cdot (e_i^p \otimes e_j^q) = (-1)^{pq} e_j^q \otimes e_i^p$$

The tensor product $R_*^{G \times G} = R_*^G \otimes_{\mathbb{Z}} R_*^G$ is a free $\mathbb{Z}[G \times G]$-resolution of $\mathbb{Z}$ with free $\mathbb{Z}[G \times G]$-generators $e_i^p \otimes e_j^q$ in degree $n = p + q$. With a free abelian group $R_n^{G \times G}$ is freely generated via $g' e_i^p \otimes g'' e_j^q$, such that $(g', g'') \in G \times G$. The action extends to an action of $C_2 \times G$ via the formula

$$(t, g) \cdot (g' e_i^p \otimes g'' e_j^q) = (-1)^{pq} g g'' e_j^q \otimes g g' e_i^p.$$

We will consider the $\mathbb{Z}[C_2 \times G]$-equivariant homomorphism

$$\phi_0 : R^{C_2} \otimes R_0^G \longrightarrow R_0^G \otimes R_0^G,$$

$$\phi_0(k^0 \otimes e_i^0) = e_i^0 \otimes e_i^0.$$

Cup$i$-product

Constructe cup-i product

The map ϕ_0 extends, using the freeness of $R^{C_2} \otimes R_*^G$ and the exactness of $R_*^G \otimes R_*^G$ to a $\mathbb{Z}[C_2 \times G]$-equivariant chain map

$$\phi_* : R_*^{C_2} \otimes R_*^G \longrightarrow R_*^G \otimes R_*^G \tag{2}$$

We now consider the cochain complex $C^*(R_*^G) = Hom_{\mathbb{Z}G}(R_*^G, \mathbb{Z})$. This notation describes a homomorphism $R_n^G \longrightarrow C^n(R_*^G),\ u \mapsto \bar{u}$. For each integer $i \geq 0$ define a $\mathbb{Z}$-linear **cup-i product**

$$C^p(R_*^G) \otimes C^q(R_*^G) \longrightarrow C^{p+q-i}(R_*^G),\ \bar{u} \otimes \bar{v} \mapsto \bar{u} \smile_i \bar{v} \tag{3}$$

by the formula

$$(\bar{u} \smile_i \bar{v})(c) = (\bar{u} \otimes \bar{v})\phi_{p+q}(k^i \otimes c) \tag{4}$$

for $c \in R_{p+q-i}^G$.

Steenrod Square

theorem

The operation

$$C^n(C_*^G) \longrightarrow C^{2n-i}(C_*^G),\ f \mapsto f \smile_i f$$

induces a homomorphism

$$Sq_i : H^n(G, \mathbb{Z}_2) \longrightarrow H^{2n-i}(G,_2).$$

The homomorphism

$$Sq^i = Sq_{n-i} : H^n(G, \mathbb{Z}_2) \longrightarrow H^{n+i}(G, \mathbb{Z}_2) \quad (5)$$

is independent of the choices in ϕ_* made in 2 and satisfies the properties of Definition 11.
We use the HAP function Mod2Steenrodalgebra(G,n) which is an implementation of Sq^i defined in 5 that inputs a finite 2-group G and a non-negative integer and returns the first $n-th$ degree of Steenrod squares.

Steenrod Square

Example

To compute the Steenrod square Sq^k for each generator and each positive 2-power $k = 2^i < degree(x)$, $x \in H^*(G_{,2})$ for $G_{32,10}$ the small group of order 32 and number 10 in GAP's library, see the following GAP session.

GAP session

$gap > G := SmallGroup(32, 10);;$
gap> A :=Mod2SteenrodAlgebra(G,8);;
gap> gens :=ModPRingGenerators(A);
$[v.1, v.2, v.3, v.4, v.6, v.9, v.15]$
gap> List(gens,A!.degree);
$[0, 1, 1, 2, 2, 3, 4]$
gap> List(gens,x->Sq(A,2,x));
$[0 * v.1, 0 * v.1, 0 * v.1, v.13, v.11, v.21, v.23 + v.24 + v.25]$
gap> PrintAlgebraWordAsPolynomial(A, List(gens,x->Sq(A,2,x))[4]);
v.4*v.4
gap> PrintAlgebraWordAsPolynomial(A, List(gens,x->Sq(A,2,x))[5]);
v.4*v.3*v.3

Steeenrod square

Definition

The Steenrod squares Sq^i of 5, defined for $i \geq 0$ are satisfying the following properties :

1. Sq^1 is the Bockstein homomorphism (denoted β in the previous chapter).
2. Sq^0 is the identity homomorphism.
3. if $deg(x) = i$ then $Sq^i(x) = x^2$.
4. if $deg(x) < i$ then $Sq^i(x) = 0$.
5. (Cartan formula) $Sq^n(xy) = \sum_{i+j=n} Sq^i(x) \smile Sq^j(y)$.
6. $Sq^i(x + y) = Sq^i(x) + Sq^i(y)$.
7. Naturality : means that for any map $f : B \longrightarrow B'$, $Sq^i(f^*) = f^*(Sq^i)$ for the cohomology homomorphism f^* induced by map f.
8. (Adem relations) $Sq^a Sq^b = \sum_{a/2}^{c=0} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$, for $a < 2b$, where $Sq^a Sq^b$ denotes the composition of the Steenrod squares and the binomail coefficient is taken modulo 2.

Steenrod square

example

To compute all Steenrod squares on $H^*(G_{,2})$, we use the formulas provided of properties in definition 11 as well as the cup product and Bockstein homomorphism, in cases where this cohomology algebra is generated by elements of degrees 1 and 2. For instance, the following commands show that for the small group $G = G_{32,4}$ of order 32 and number 4 in the small group library of the computer algebra system GAP, the image of the homomorphism

$$Sq^3 : H^4(G_{32,4,2}) \longrightarrow H^7(G_{32,4,2})$$

is a vector space of dimension 1 generated by xv^3; here the algebra $H^*(G_{32,4,2})$ is generated by four elements x, y of degree 1 and z, v of degree 2.

Steenrod square

GAP session

```
gap > G := SmallGroup(32, 4); ;
gap> A :=ModPSteenrodAlgebra(G,7) ; ;
gap> H4 :=Filtered(Basis(A),x->A !.degree(x)=4) ; ;
gap> Sq3H4 :=List(H4,x->Sq(A,3,x)) ; ;
gap> Dimension(Submodule(A,Sq3H4)) ;
1
gap> B :=Basis(Submodule(A,Sq3H4)) ; ;
gap> PrintAlgebraWordAsPolynomial(A,B[1]) ;
v.6*v.6*v.6*v.2
```

Steenrod square

computing Steenrod square

Also we using the HAP command CohomologicalData(G,n) to determine and print details of the group order, group number, cohomology ring generators with degree and relations and the Steenrod square Sq^k for each generator x and each positive 2-power $k = 2^i < degree(x)$. If we want the cohomology ring details printed to a file then this file name is included as an optional third input to the command. Also the command CohomologicalData(G,n) returns the following information for $n = 6$ and $G_{32,30}$ the small group of order 32 and number 30 in GAP's library (see **??** and **??**). It prints correct information for the cohomology ring $H^*(G,_2)$ of a 2-group G provided that the integer n is at least the maximal degree of a relator in a minimal set of relators for the ring, moreover n trems of a free $\mathbb{F}G$-resolution is enough to compute the whole mod-2 cohomology ring by the tables of King and Green.

Steenrod square

example

Group order : 32
Group number : 30
Group description : (C4 x C2 x C2) : C2
Cohomology generators
Degree 1 : a, b, c
Degree 2 : d
Degree 3 : e, f
Degree 4 : g
Steenrod squares
$Sq^1(d) = d * a + d * c$
$\mathrm{Sq}^1(e) = d * a * b + e * b$
$\mathrm{Sq}^2(e) = d * a * a * a + d * b * b * c + d * d * c + g * a$
$\mathrm{Sq}^1(f) = d * b * b + d * b * c + e * b$
$\mathrm{Sq}^2(f) = d * d * c + f * b * b + g * c$
$\mathrm{Sq}^1(g) = d * a * a * a + d * b * b * c + e * a * a$
$\mathrm{Sq}^2(g) = d * a * a * a * a + d * b * b * b * b + d * d * a * a + d * d * a * b + d * d * b * b + d * d * b * c + e * a * a * a + e * b * b * b + f * f + g * b * b$

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