Introduction to Alternating Sign Matrices

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The number of $n \times n$ *ASMs* is $\frac{1!4!7!\ldots(3n-2)!}{n!(n+1)!(n+2)!\ldots(2n-1)!}$.

The λ-determinant

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These are the only two 2×2 *ASMs*.

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For a 3×3 matrix A, the λ-determinant of A is

$$a_{11}a_{22}a_{33} + \lambda(a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}) + \lambda^2(a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31})$$

$$+\lambda^3 a_{13}a_{22}a_{31} + (\lambda + \lambda^2)a_{22}^{-1}a_{12}a_{21}a_{23}a_{32}$$

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These terms correspond to the seven 3×3 *ASMs*:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

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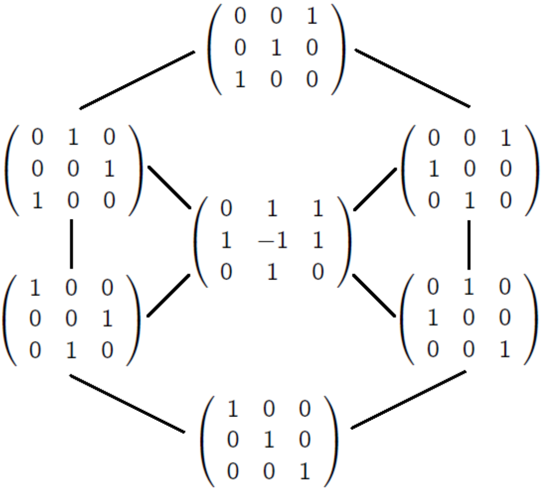
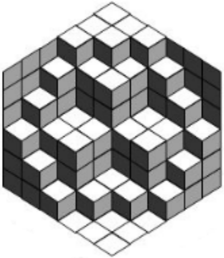
Bruhat Order

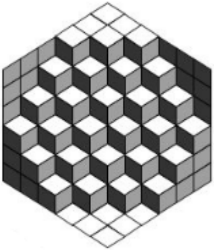


Totally Symmetric Self-Complementary Plane Partitions

A *TSSCPP* is a partition of the number $4n^3$ which can be thought of as follows:

> A stack of $1 \times 1 \times 1$ cubes in a $2n \times 2n \times 2n$ box, pushed into a corner (the origin), so that
> - The stack is symmetric under all permutations of the three coordinates
> - The complement is a copy of the stack itself





Tilings

There is also a link between *ASMs* and tilings of *gaskets* and *baskets*.

Figure: A gasket and a basket

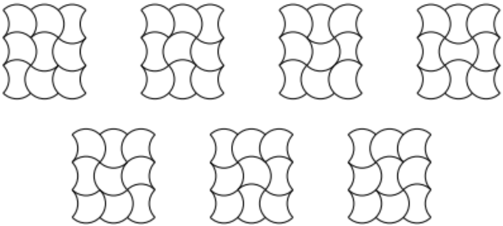


Figure: The seven possible 3×3 tilings correspond to the seven 3×3 *ASMs*

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- The *pattern* of an *ASM* A is $\tilde{A} = A_1 + A_2$
- We define R to be the vector of length n where the ith entry is equal to the sum of the ith row of $\tilde{A}$. Similarly, we define S to be the vector of length n where the ith entry is equal to the sum of the ith column of $\tilde{A}$.

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- Both bounds are achievable, with the permutation matrices achieving the lower bound, and the *diamond ASMs* achieving the upper bound.

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D_4:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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D_5:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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