

Noise and Multistability in the Square Root Map

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Noise and Nonsmoothness in Dynamical Systems

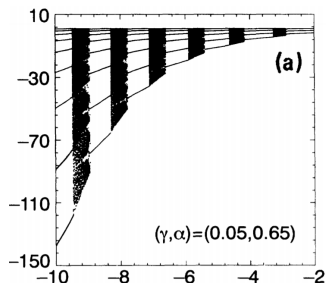
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- Nonsmooth systems - qualitative changes in the behavior of the system under parameter variation that do not occur in the smooth setting.
- Adding noise to (smooth) systems - does more than just blur the outcome of the system in the absence of noise

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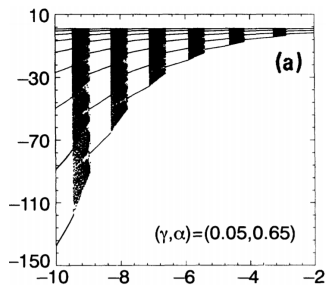


Figure: From [CONG94].

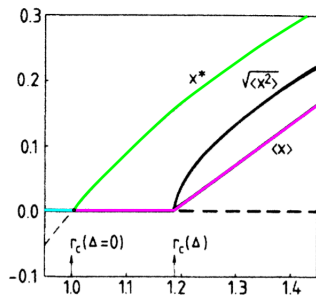


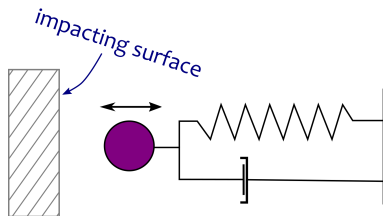
Figure: Adapted from [LL86].

The Square Root Map

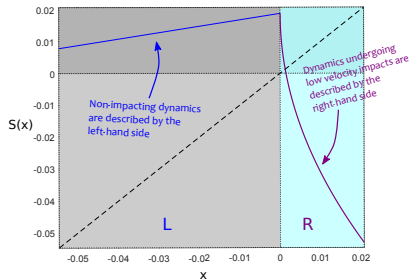
Many impacting systems, including rattling gears, moored boats impacting docks, Braille printers, percussive drilling and atomic force microscopes are described by a 1-D map known as the square root map near *grazing* impacts.

$$x_{n+1} = S(x_n) = \begin{cases} +bx_n & \text{if } x_n < 0; \\ a\sqrt{-x_n} & \text{if } x_n \geq 0; \end{cases}$$

where $a > 0$ and $b > 0$.

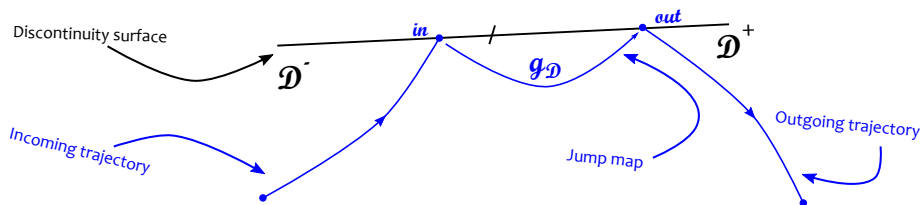


A forced impact oscillator



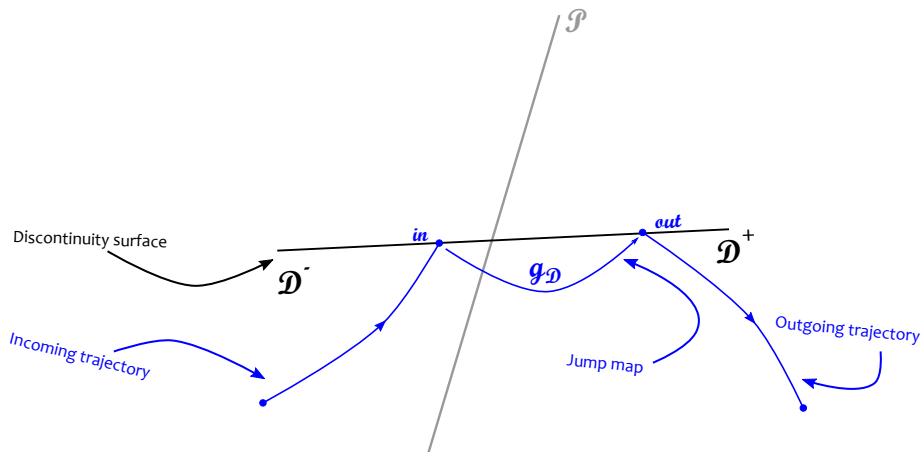
Symbolically, if $x_n < 0$ it is represented by an L and if $x_n > 0$ it is represented by an R .

Deriving The Square Root Map



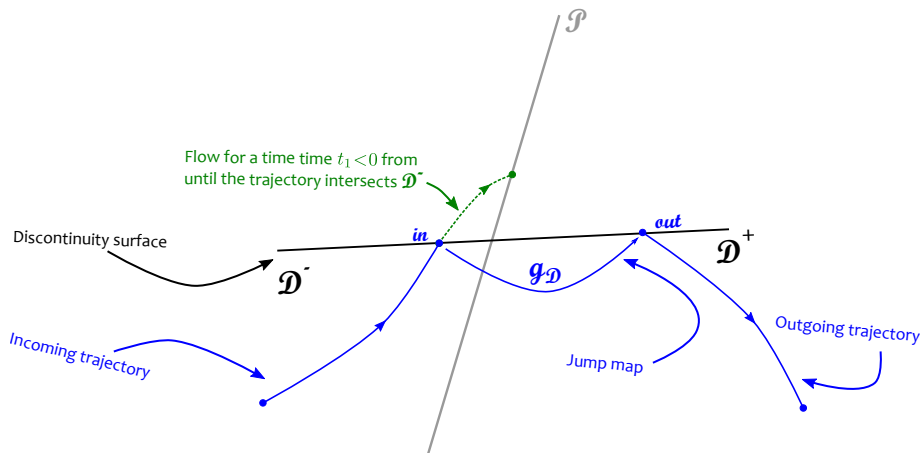
Deriving the map from the full system

Deriving The Square Root Map



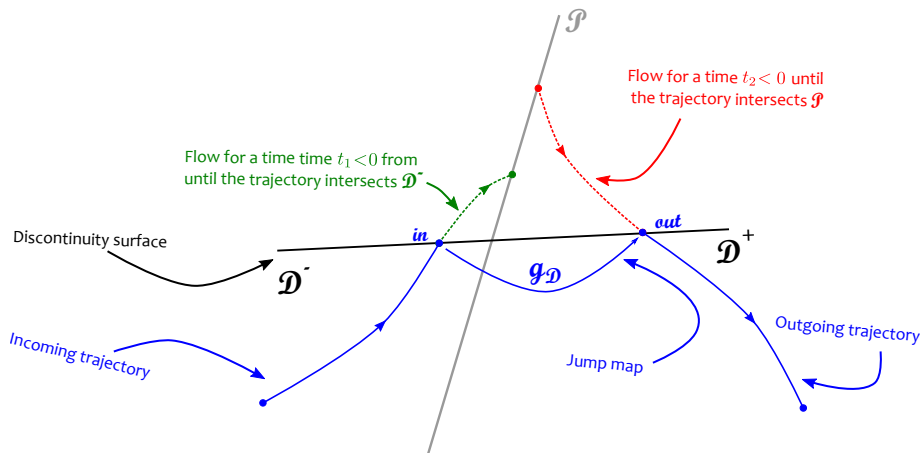
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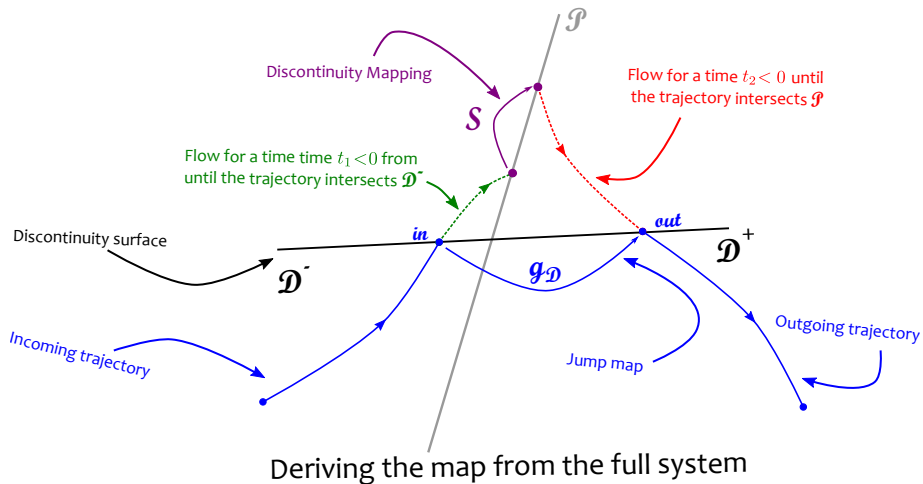
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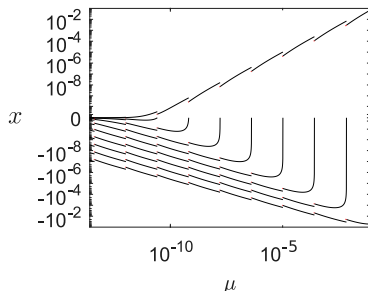
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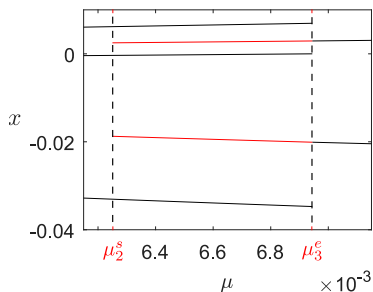
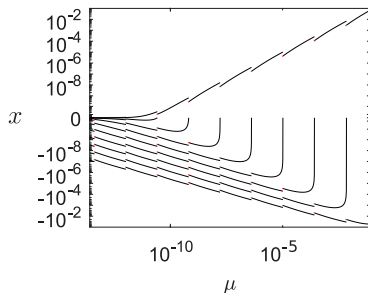
The Period Adding Cascade

Here we will assume that the parameter b (the slope of the linear part) is such that $0 < b < 1/4$. For values of b in this range the deterministic square root map undergoes a period-adding cascade with intervals of bistability as the bifurcation parameter μ is decreased.



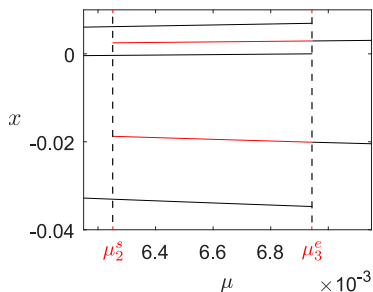
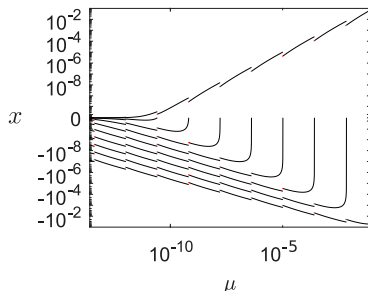
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These periodic orbits take the form $(RL^m)^1$ for $m = 1; 2; 3; \dots$. This means they consist of one iterate on the right (> 0) followed by m iterates on the left (< 0).

Riddled Basins of Attraction

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The Square Root Map With Additive Noise

In [SHK13] Simpson, Hogan and Kuske show that white noise in the piecewise smooth flow translates to additive white noise in the square root map. This noise formulation may be sensible to model systems where the forcing term or external fluctuations represent a significant source of uncertainty.

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The square root map with additive Gaussian white noise is given by

$$x_{n+1} = S_a(x_n) = \begin{cases} bx_n + \eta_n & \text{if } x_n < 0 \\ a\sqrt{|x_n|} + \eta_n & \text{if } x_n \geq 0; \end{cases} \quad (1)$$

where η_n are identically distributed independent normal random variables with mean 0 and standard deviation σ , $\eta_n \sim N(0; \sigma^2)$.

Noisy Bifurcation Diagrams

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Noise Amplitude and Proportions of Periodic Behaviour

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Inducing Bistability

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In the numerical simulations we have found that noise-induced transition from period-3 to period-2 behaviour in regions where period-2 behaviour unstable display certain similarities. In particular, we have observed that the transitions tend to take the following symbolic form

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$$\text{RLLRLL} :: \text{RLLRLRLRLRL} :: \text{RLRL} : \quad (2)$$

The significant feature of the symbolic representation of the transition above is the repeated R, corresponding to repeated iteration on the right-hand side of the square root map, i.e. repeated low-velocity impact in the physical system.

Noise and Deterministic Structures

We note that the set of initial values that are on the right which remain on the right after iteration by the deterministic square root map are given by the interval

$$A_{RR} = [0; (a)^2] : \quad (3)$$

We also note that the last left iterate of the period-3 orbit is very close to 0 for values of a close to the interval of multistability.

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Therefore, it is not hard to see that noise has the potential to push the last left iterate of a period-3 orbit into A_{RR} inducing repeated R's or repeated grazing impacts.

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Generalising to Higher Periodicities

The features of this transition are repeated as we look at transitions from RL^m behaviour to RL^{m-1} behaviour for increasing m . In particular we observe transitions of the form

$$RL^m RL^m \dots RL^m \underline{RL^{m-1} RL^{k-2} RL^{m-1}} RL^{m-1} \dots RL^{m-1} \quad (4)$$

for ϵ in a neighbourhood of $\frac{s}{m}$ such that $\epsilon < \frac{s}{m}$ and $k \in \{2, 3, \dots, m\}$.

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Conclusions

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This behaviour can be generalised to higher periodicities.

The effect of the addition of noise on intervals of multistability of increasing minimal periodic orbit obeys a scaling law.







SR Bishop, Impact oscillators *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* **347** (1994), no. 1683, 347{351.

W. Chin, E. Ott, H. E. Nusse, and C. Grebogi, Grazing bifurcations in impact oscillators *Physical Review E* **50** (1994), no. 6, 4427{4444.

M. di Bernardo, C. J. Budd, A. R. Champneys, and P. Kowalczyk, Piecewise-smooth dynamical systems: Theory and applications *Applied Mathematical Sciences*, vol. 163, Springer-Verlag London Ltd., London, 2008.

John Guckenheimer and Philip J Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, vol. 42, Springer Science & Business Media, 2013.

S. J. Linz and M. Lucke, Effect of additive and multiplicative noise on the first bifurcations of the logistic model *Physical Review A* **33** (1986), no. 4, 2694{2703.

-  A.B. Nordmark, *Non-periodic motion caused by grazing incidence in an impact oscillator*, J. Sound Vib. **145** (1991), 279–297.
-  _____, *Universal limit mapping in grazing bifurcations*, Phys. Rev. E **55** (1997), 266–270.
-  D.J.W. Simpson, S.J. Hogan, and R. Kuske, *Stochastic regular grazing bifurcations*, SIADS **12** (2013), 533–559.
-  D.J.W. Simpson and R. Kuske, *The influence of localised randomness on regular grazing bifurcations with applications to impact dynamics*, Journal of Vibration and Control **12** (2016), 1077546316642054.
-  E.J. Staunton and P.T. Piiroinen, *The effects of noise on multistability in the square root map*, Physica D (2018), In Press.
-  _____, *Noise induced multistability in the square root map*, Under Review (2018), TBC.