# Noise and Bistability in the Square Root Map

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4 September 2018

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# Noise and Nonsmoothness in Dynamical Systems

Both noise and nonsmoothness have been shown to independently be the drivers of significant changes in qualitative behaviour.

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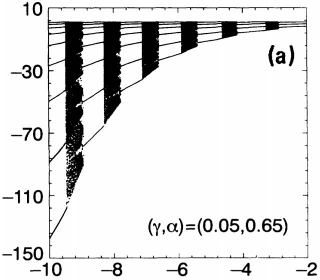

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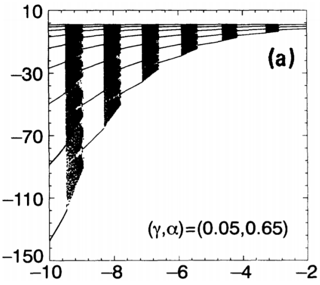



Figure: From [CONG94].

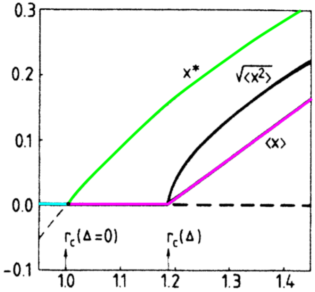


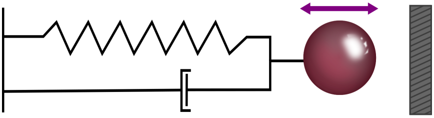

Figure: Adapted from [LL86].

# The Square Root Map

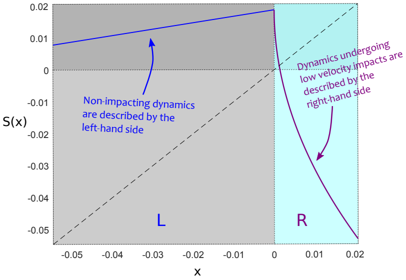
Many impacting systems, including rattling gears, moored boats impacting docks, Braille printers, percussive drilling and atomic force microscopes are described by a 1-D map known as the square root map near *grazing* impacts.

$$x_{n+1} = S(x_n) = \begin{cases} + bx_n & \text{if } x_n < 0; \\ a\sqrt{x_n} & \text{if } x_n \quad 0; \end{cases}$$

where $a > 0$ and $b > 0$.



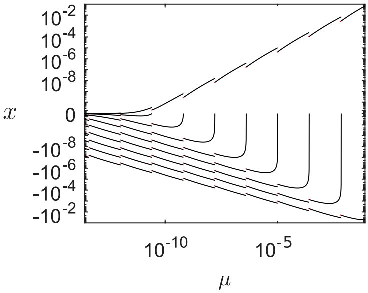
A forced impact oscillator.




Symbolically, if $x_n < 0$ it is represented by an $L$ and if $x_n > 0$ it is represented by an $R$.

# The Period Adding Cascade

Here we will assume that the parameter $b$ (the slope of the linear part) is such that $0 < b < 1{=}4$. For values of $b$ in this range the deterministic square root map undergoes a period-adding cascade with intervals of bistability as the bifurcation parameter is decreased.



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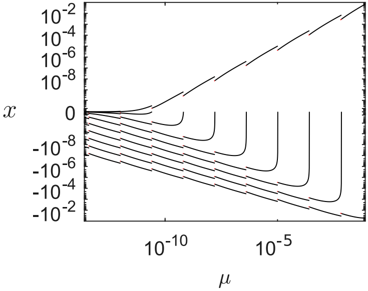


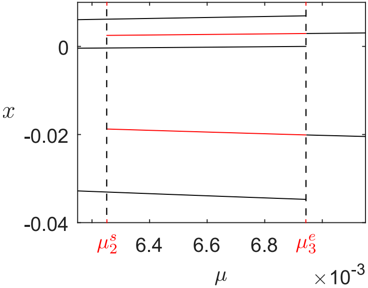

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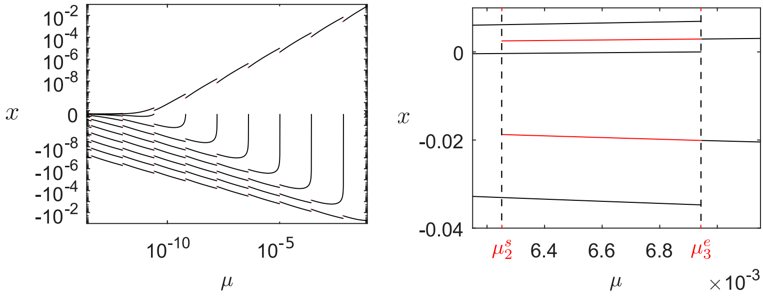


These periodic orbits take the form $(RL^m)^1$ for $m = 1; 2; 3; : : :$. This means they consist of one iterate on the right ($> 0$) followed by $m$ iterates on the left ($< 0$).

# Riddled Basins of Attraction

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# The Square Root Map With Additive Noise

In [SHK13] Simpson, Hogan and Kuske show that white noise in the piecewise smooth  ow translates to additive white noise in the square ro map. This noise formulation may be sensible to model systems where t forcing term or external  uctuations represent a signi cant source of uncertainty.

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The square root map with additive Gaussian white noise is given by

$$x_{n+1} = S_a(x_n) = \begin{cases} \; + bx_n + \;_n & \text{if } x_n < 0 \\ a\sqrt{x_n} + \;_n & \text{if } x_n \quad 0; \end{cases} \qquad (1)$$

where $_n$ are identically distributed independent normal random variables with mean 0 and standard deviation , $_n \quad N(0; \; ^2)$.

# Noisy Bifurcation Diagrams

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# Noise Amplitude and Proportions of Periodic Behaviour

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# Inducing Bistability

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In the numerical simulations we have found that noise-induced transitior from period-3 to period-2 behaviour in regions where period-2 behaviou unstable display certain similarities. In particular, we have observed that the transitions tend to take the following symbolic form

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The signi cant feature of the symbolic representation of the transition above is the repeated R, corresponding to repeated iteration on the right-hand side of the square root map, i.e. repeated low-velocity impact in the physical system.

# Noise and Deterministic Structures

We note that the set of initial values that are on the right which remain on the right after iteration by the deterministic square root map are giver by the interval

$$A_{RR} = \quad 0; ( \; =a \,)^{2} \quad : \qquad (3)$$

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We also note that the last left iterate of the period-3 orbit is very close to 0 for values of   close to the interval of bistability.

Therefore, it is not hard to see that noise has the potential to push the last left iterate of a period-3 orbit into $A_{RR}$ inducing repeated R's or repeated grazing impacts.

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# Generalising to Higher Periodicities

The features of this transition are repeated as we look at transitions from $RL^{m}$ behaviour to $RL^{m\ \ 1}$ behaviour for increasing $m$. In particular we observe transitions of the form

$$RL^{m}RL^{m}:::RL^{m}\underline{RL^{m\ \ 1}RL^{k\ \ 2}RL^{m\ \ 1}}RL^{m\ \ 1}::::::RL^{m\ \ 1} \qquad (4)$$

for in a neighbourhood of $^{s}_{m}$ such that $<$ $^{s}_{m}$ and $k$ 2 f 2; 3; : : : ; $m$g.

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for $\mu$ in a neighbourhood of $\mu^s_m$ such that $\mu < \mu^s_m$ and $k \in \{2, 3, \ldots, m\}$.
The most significant feature of this transition is the sequence $RL^{k-2}R$ for $k \in \{2, 3, \ldots, m\}$, corresponding to iterations on the right-hand side of the map being repeated more quickly than is usual for a settled system with $\mu < \mu^s_m$.

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- This behaviour can be generalised to higher periodicities.
- The effect of the addition of noise on intervals of bistability of increasing minimal periodic orbit obeys a scaling law.

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