Noise and Bistability in the Square Root Map

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Abstract—Recurrent mechanical real-world systems with impacts are often modelled using impact oscillators. Near low-velocity impacts the dynamics of impact oscillators can be described by a one-dimensional map known as the square root map. Here we will describe the complex structure of the basins of attraction of stable periodic orbits of the square root map and how this produces sensitivity to the addition of small-amplitude white noise. In particular we will focus on the effects of noise of varying amplitudes on the square root map close to parameter values that lead to bistability.

1. Introduction

Traditionally smooth deterministic dynamical systems are used to model real-world phenomena. These models present a simplified view of real-world systems where, on one hand, the evolution of systems is always smooth and exhibits no interruptions such as impacts, switches, slides or jumps and, on the other hand, the future of any system is completely determined by its present state with no uncertainty (or noise) [11]. However, independently, both nonsmoothness and noise have been shown to be the drivers of significant changes in qualitative behaviour. In nonsmooth systems we find certain types of qualitative changes in the behaviour of the system, known as *discontinuity induced bifurcations*, that do not occur in the smooth setting [2, 7]. Adding noise to smooth but nonlinear systems has been shown to have the potential to do far more than just blur the outcome of the system in the absence of noise, especially close to bifurcation points [9, 10, 16]. As a result, it is therefore of particular interest to investigate and understand how the inclusion of noise can effect the qualitative dynamics of a nonsmooth system close to discontinuity-induced bifurcations.

An *impact oscillator* is a forced mechanical system that undergoes impacts at rigid stops. Many real-world mechanical systems including systems arising in engineering, for instance moored ships impacting a dock or rattling gears are modelled using impact oscillators [3]. Due to the presence of rigid impacts in such systems they are best modelled using nonsmooth dynamical systems. It is important to understand such systems in order to avoid problems, such as wear and noise. In particular, since real-world systems, including mechanical systems, are subject to uncertainties, we must also investigate how stochastic noise can affect such systems. In the case of impact oscillators, noise could for instance arise due to background vibrations or measurement errors.

In this paper we will investigate the effects of the additive noise on the qualitative behaviour of a piecewise-smooth map known as the *square root map* [1, 4, 12, 13, 14]. The map can be derived as an approximation for solutions of a piecewise-smooth ordinary differential equation describing the dynamics of an *impact oscillator* near *grazing* (low-velocity) impacts [12, 15] and it exhibits non-standard qualitative behaviour as a result of a discontinuity in its first derivative. In particular we will focus on the effect of the introduction of noise near bifurcation points in the *period-adding cascade* of the square root map, a bifurcation structure which is unique to nonsmooth systems.

2. Bistability in the Square Root Map

We will consider the one-dimensional square root map

$$x_{n+1} = S(x_n) = \begin{cases} S_L(x_n) = \mu + bx_n, & x_n < 0, \\ S_R(x_n) = \mu - a\sqrt{x_n}, & x_n \geq 0, \end{cases} \qquad (1)$$

where $S_L(x)$ is the linear part of the map applied on the left-hand side when $x < 0$, and $S_R(x)$ is the square root part applied on the right when $x \geq 0$. Here we will assume that $a > 0$ and $0 < b < \frac{1}{4}$. For values of b in this range the deterministic square root map undergoes a period-adding cascade with intervals of bistability as the bifurcation parameter μ is decreased [13], this structure can be clearly seen in Figure 1a).

We see that there are values of $\mu > 0$ for which a stable periodic orbit of period m with exists for each $m = 2, 3, \ldots$, and other values of $\mu > 0$ such that there are two stable periodic orbits coexisting, one of period m and the other of period $m + 1$. These are the only possible attractors except at bifurcation points. We denote the interval of μ values for which a period-m orbit exists as an attractor (μ_m^s, μ_m^e).

Note that all attractors have symbolic codes of the form $(RL^n)^\infty$ meaning that they have exactly one iterate on the right (R) and the remaining n iterates are on the left (L).

3. The Stochastic Square Root Map

Hogan, Simpson and Kuske [17] have shown that the square root map in two dimensions with additive Gaussian white noise arises when the source of uncertainty in

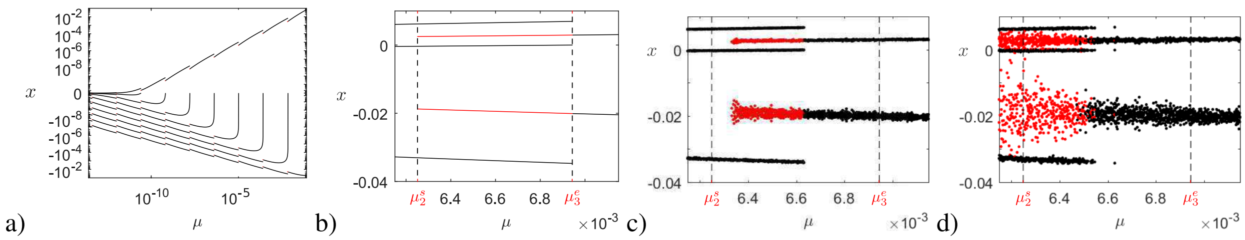

Figure 1: Bifurcation diagrams for the deterministic square root map, S, with $a = 0.5$, $b = 0.2$. a) The period adding cascade of attractors $(RL^m)^\infty$ for $m \in \{1, \ldots, 10\}$. On the intervals of μ where $(RL^{m-1})^\infty$ and $(RL^m)^\infty$ coexist as attractors the iterates of $(RL^{m-1})^\infty$ are marked in red. A symmetric logarithmic transformation [20] has been applied to the x-axis in order to clearly show the structure of the period adding cascade. b) The coexistence of attractors $(RL)^\infty$ and $(RLL)^\infty$ for μ about the interval (μ_2^s, μ_3^e). The period-2 $(RL)^\infty$ orbit is coloured red on the interval of bistability. c) & d) Bifurcation diagrams for the square root map with additive Gaussian white noise (2) for μ in a neighbourhood of the coexistence interval (μ_2^s, μ_3^e). The deterministic values of μ_2^s and μ_3^e are indicated by dashed lines. Where the two periodic behaviours coexist the iterates of period-2 are marked in red. In c) the noise amplitude $\Delta = 4 \times 10^{-5}$ while in d) $\Delta = 1 \times 10^{-4}$

the full system is practically independent of the state of the system. With this in mind, we will consider small amplitude, additive, Gaussian white noise in the one-dimensional square root map. The square root map with additive Gaussian white noise is given by

$$x_{n+1} = S_a(x_n) = \begin{cases} \mu + bx_n + \xi_n, & x_n < 0, \\ \mu - a\sqrt{x_n} + \xi_n, & x_n \geq 0, \end{cases}$$
$$\xi_n \sim N(0, \Delta^2), \qquad (2)$$

where ξ_n are identically distributed independent normal random variables with mean 0 and standard deviation or amplitude Δ.

4. Numerical Observations

The effect of noise on the dynamics of a system with multiple coexisting attractors has long been of interest [5, 6, 8]. In this paper we focus on phase-space sensitivity for values of the bifurcation parameter μ close to intervals where period-m and period-$(m + 1)$ attractors coexist [18, 19]. We see in Figure 1 b)-d) that the relationship between noise amplitude and the behaviour of the system in these regions is complex and non-monotonic.

For example, for fixed μ close to μ_2^s it appears that with increasing noise amplitude we will first see a decrease in the probability of being in period-2 behaviour to some minimum followed by an increase in this probability as Δ increases further. We can confirm this relationship by examining Figure 2 which shows the changing proportion of iterates spent by 1000 orbits with linearly spaced initial conditions in each of the two periodic behaviours after discarding transients when $\mu = 0.00637$. We see that beyond some threshold noise amplitude there is a significant shift in the proportion of iterates from period-2 behaviour to period-3 behaviour until almost all behaviour is period-3, however increasing the amplitude further we see a return of period-2 behaviour.

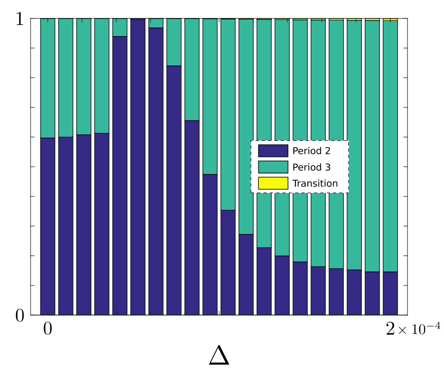

Figure 2: Bar chart showing the changing proportion of time spent in period-2 and period-3 behaviour for increasing amplitude of additive noise Δ, where $a = 0.5$, $b = 0.2$ and $\mu = 0.00637$. The amplitude Δ ranges from 0 (deterministic square root map) to 2×10^{-4}.

5. Results

The relationships between noise amplitude and periodic behaviour observed are highly dependent on the bifurcation parameter μ. In order to explain the observed relationships we examine how noise interacts with the deterministic structures of the map. In particular, we examine how approximations for the distribution of steady-state trajectory deviations resulting from the addition of noise to the system derived in [18] can be related to the basins of attraction of coexisting attractors in regions of bistability, and to relative levels of contraction and expansion experienced by trajectories close to, but outside these regions.

In regions of multistability we focus on three features of the relationship between the noise amplitude Δ and the proportion of iterates spent in period-m and period-$(m + 1)$ behaviour for a given value of μ. First, we identify the minimum noise amplitude required to induce a *significant shift* in the proportions spent in either behaviour compared to the deterministic case. Next, we identify the noise amplitude required for the *effective destruction* of one of the periodic attractors, i.e. the noise amplitude required to have less than .01 per cent of iterates spent in the destroyed behaviour. Finally we identify the noise amplitude required for the *reversal of relationships* between the noise amplitude and the proprtions, i.e. if increasing noise amplitude initially resulted in an increase (decrease) in the proportion of iterates spent in period-m behaviour this is the noise amplitude required for this proportion to decrease (increase) once more.

Let ϱ denote the ratios between the minimum distance from each iterate of the periodic orbits to the boundary of their corresponding basins of attraction and the standard deviation of the steady state deviation distribution associated with that iterate. In Figure 3 we plot the threshold ϱ ratios associated with each of the three features we are interested in. We observe that at each end of the interval each feature is associated with an approximately constant value of ϱ for the attractor whose proportion is decreasing. This shows the importance of the interaction between the steady-state deviation distributions and the deterministic structures of the map, namely its basins of attraction. In a small region a higher noise amplitude and hence a lower ϱ ratio is required to induce each of the three features. This is due to the fact that in this small region the noise amplitude required to push orbits out of their basins of attraction at a significant rate is similar for both attractors. As a result an even higher amplitude is required for one effect to dominate the other and produce a significant overall change in proportions.

Outside regions of multistability we focus on the potential for noise to induce transitions from period-$(m+1)$ to period-m behaviour in regions close to μ_m^s but where $\mu < \mu_m^s$, i.e. in regions where the period-m orbit is unstable and the period-$(m+1)$ orbit is a global attractor in the deterministic system. We find that such transitions take the symbolic form

$$RL^mRL^m\cdots RL^m\underline{RL^{k-2}R}L^{m-1}RL^{m-1}\cdots RL^{m-1}, \qquad (3)$$

where R denotes an iterate on the right, L denotes an iterate on the left and $2 \leq k \leq m$. The most significant feature of this transition is the underlined portion $RL^{k-2}R$ which represents repeated iteration on the right over a smaller number of iterates than in period-$(m+1)$ behaviour or in period-m behaviour. Here period-$(m+1)$ behviour is the only stable behaviour in the deterministic system for this value of μ and period-m behaviour is stable for nearby values of μ. We find that the initial conditions which produce such a sequence under iteration by the squre root map (1) are the intervals

$$A_{RR} = \left(0, \left(\frac{\mu}{a}\right)^2\right) \qquad (4)$$

for $k = 2$ and

$$A_{RL^{k-2}R} = \left(\left(\frac{\mu}{a}\sum_{i=0}^{k-3} b^{-i}\right)^2, \left(\frac{\mu}{a}\sum_{i=0}^{k-2} b^{-i}\right)^2\right) \qquad (5)$$

for $k \in \{3, 4, \ldots, m\}$. These sets are located just to the right of zero, as a result, a small positive deviation due to low amplitude noise could push settled RL^m dynamics into one of these sets. The image of these intervals after the sequence $RL^{k-2}R$ are concentrated around the first left iterate of the unstable period-m orbit due to repeated iteration by the square root part of the map on the right over a small number of iterates. Orbits with initial conditions close to the unstable period-m orbit can take a significant number of iterates to transition back to period-$(m+1)$ behaviour in the deterministic system and indeed in the system with low amplitude noise. As a result we can see why noise induced transitions from period-$(m+1)$ to period-m behaviour take the form given in (3).

6. Conclusion

Our approach shows us the importance of the relationships between the deterministic structures of the square root map that arise from its nonsmoothness and steady state deviation distributions in determining the overall effect of the addition of low amplitude noise. In particular, examining the period-adding bifurcation cascade and the structure of the basins of attraction of the deterministic square root map allows us to understand the relationships between noise amplitude and periodic behaviour observed on intervals of bistability while we also see the importance of the square root singularity to noise-induced transitions to unstable periodic behaviour outside such intervals. We find that there is a complicated nonmonotonic relationship between noise amplitude and the proportion of iterates spent in each periodic behaviour on intervals of bistability and that noise can effectively stabilise unstable periodic behaviour outside these intervals.

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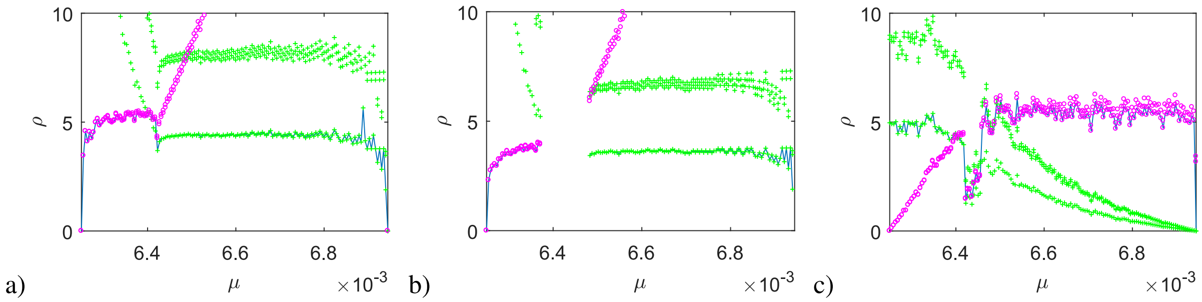

Figure 3: The ϱ values associated with the noise amplitudes required to a) cause a significant change in the proportion of time spent in the two periodic behaviours, b) destroy periodic behaviour, and c) induce a reversal in the proportion changes (final panel), for $\mu \in (\mu_2^s, \mu_3^e) \approx (0.00625, 0.00694)$. Each of the ϱ values of the period-2 orbits are marked with a magenta $\circ$ while each of the ϱ values of the period-3 orbits are marked with a green $+$. In all three cases the blue line traces the minimum ϱ ratio of the attractor whose proportion is diminishing in order to create the feature of the relationship in question.

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