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Linearising Nonsmooth Flows with Stochastic Discontinuity Boundaries

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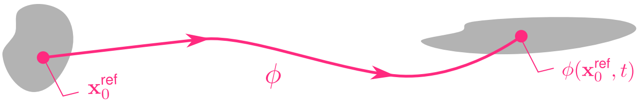
Linearisation



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$$\underline{x} = f(x); \quad x = x_0; \quad \text{has the unique solution} \quad x(t) = \quad (x_0; t). \tag{1}$$

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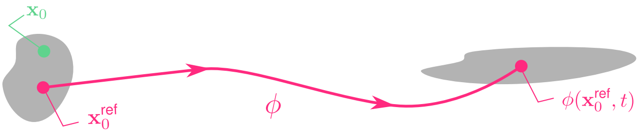

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Then for x_0 in a small neighbourhood of x_0^{ref}

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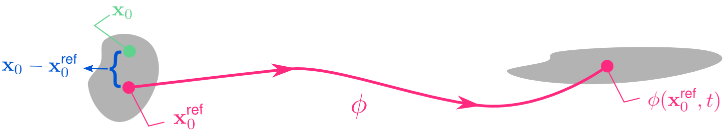

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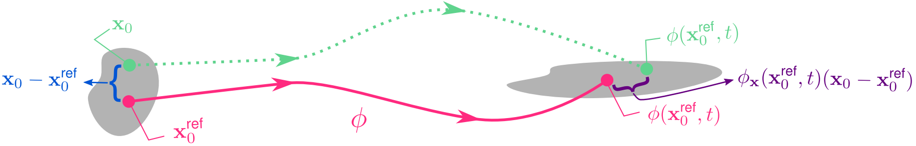

Suppose the system

$$\dot{x} = f(x); \quad x(0) = x_0; \quad \text{has the unique solution} \quad x(t) = \phi(x_0; t). \tag{1}$$

Then for x_0 in a small neighbourhood of x_0^{ref}

$$\phi(x_0; t) - \phi(x_0^{\text{ref}}; t) = \Phi_{\mathbf{x}}(x_0^{\text{ref}}; t)(x_0 - x_0^{\text{ref}}) + O(\|x_0 - x_0^{\text{ref}}\|); \tag{2}$$

where the Jacobian $\Phi_{\mathbf{x}}(x_0^{\text{ref}}; t)$ is the solution to the IVP

$$\dot{\Phi} = f_{\mathbf{x}}(\phi(x_0^{\text{ref}}; t))\Phi; \qquad \Phi(0) = \mathbf{I}. \tag{3}$$

Linearisation

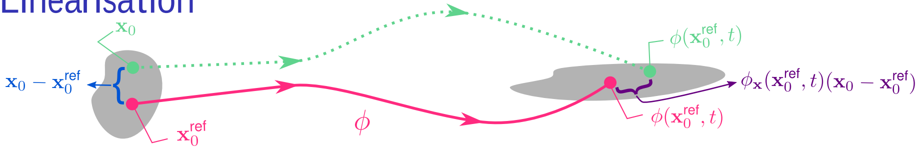

Figure: Linearisation of smooth dynamical systems

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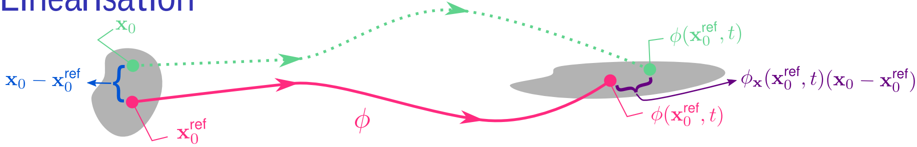

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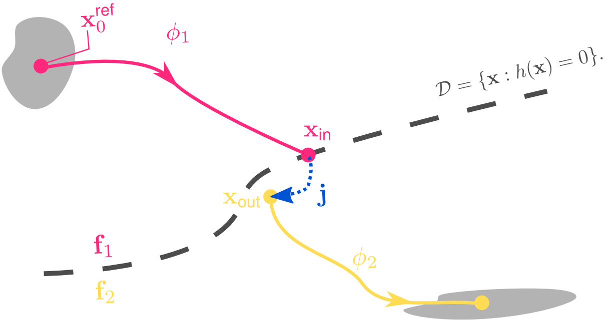

Figure: A nonsmooth dynamical system

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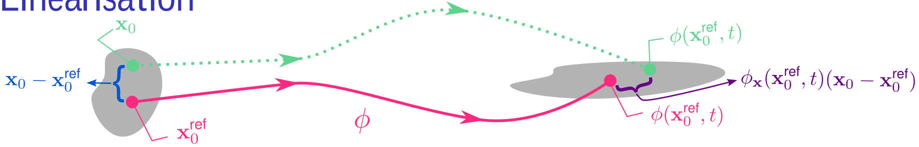

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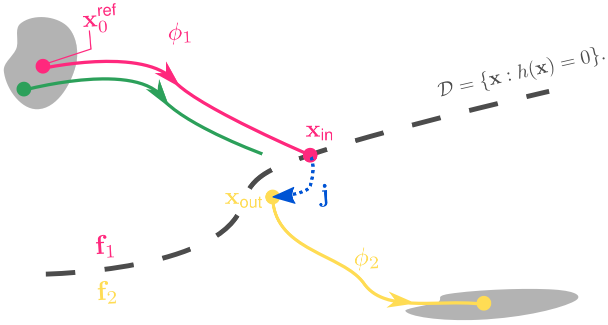

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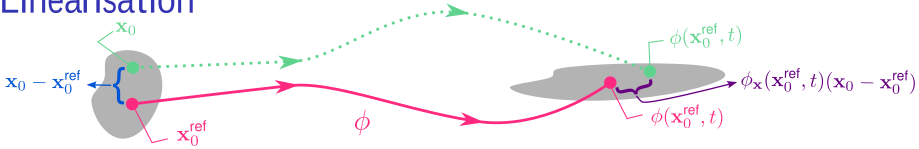

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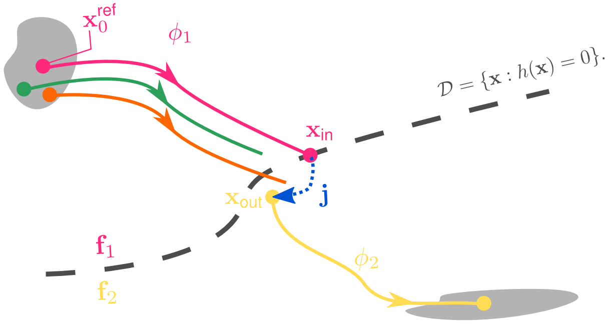

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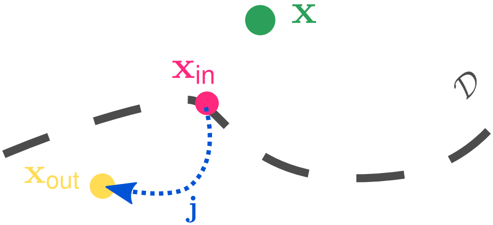
Constructing the Zero-Time Discontinuity Mapping



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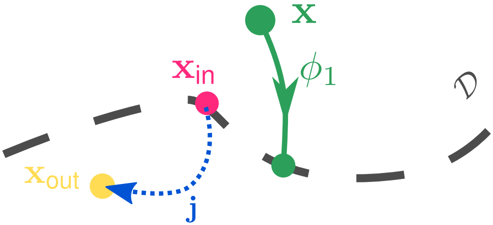

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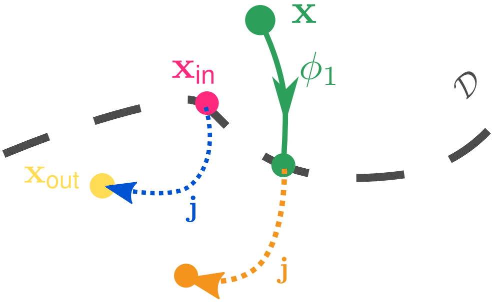

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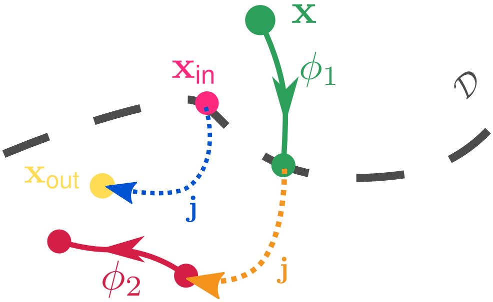

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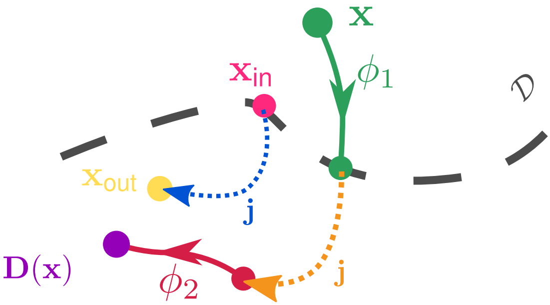

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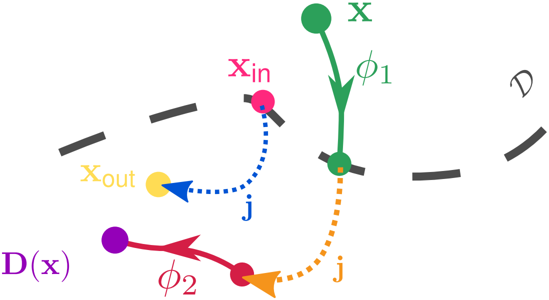

Figure: Constructing the ZDM

We can now write

$$(\mathrm{x}_0;T) = {}_2(\mathrm{D}({}_1(\mathrm{x}_0;t_{\mathrm{ref}}));T \quad t_{\mathrm{ref}}); \tag{4}$$

where the ZDM

$$\mathrm{D}(\mathrm{x}) = {}_2(\mathrm{j}({}_1(\mathrm{x};t(\mathrm{x})));\quad t(\mathrm{x})) \tag{5}$$

takes a point in a neighbourhood of x_{in} and maps it to a point in a neighbourhood of $\mathrm{x}_{\mathrm{out}}$.

The Saltation Matrix

The Jacobian of D evaluated at x_{in} is given by

$$\begin{aligned} \mathrm{D}_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}}) &= \mathrm{j}_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}}) + (\mathrm{j}_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}}) \quad \mathrm{f}_{\mathrm{out}}) t_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}}) \\ &= \mathrm{j}_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}}) + \frac{(\mathrm{f}_{\mathrm{out}} \quad \mathrm{j}_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}})\mathrm{f}_{\mathrm{in}}) h_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}})}{h_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}})\mathrm{f}_{\mathrm{in}}}; \end{aligned} \tag{6}$$

where $\mathrm{f}_{\mathrm{in}} = \mathrm{f}_1(\mathrm{x}_{\mathrm{in}})$ and $\mathrm{f}_{\mathrm{out}} = \mathrm{f}_2(\mathrm{x}_{\mathrm{out}})$. In the case where h is explicitly time-dependent this becomes

$$\mathrm{D}_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}}) = \mathrm{j}_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}}) + \frac{(\mathrm{f}_{\mathrm{out}} \quad \mathrm{j}_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}})\mathrm{f}_{\mathrm{in}}) h_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}}; t_{\mathrm{ref}})}{h_t(\mathrm{x}_{\mathrm{in}}; t_{\mathrm{ref}}) + h_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}}; t_{\mathrm{ref}})\mathrm{f}_{\mathrm{in}}}: \tag{7}$$

In both cases we have that

$$_{\mathbf{x}}(\mathrm{x}_0^{\mathrm{ref}}; T) = \; _{2;\mathbf{x}}(\mathrm{x}_{\mathrm{out}}; T \quad t_{\mathrm{ref}}) \mathrm{D}_{\mathbf{x}}(\mathrm{x}_{\mathrm{in}}) \; _{1;\mathbf{x}}(\mathrm{x}_{\mathrm{in}}; t_{\mathrm{ref}}): \tag{8}$$

Introducing Noise

As in the deterministic case, we define the discountinuity boundary D as the zeros of a function h. For a stochastically oscillating boundary we let h take the form

$$h(x;t) = \hat{h}(x;t) \quad P(t); \tag{9}$$

where the function $\hat{h}$ is deterministic and $P(t)$ is a stochastic process.

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Let $\hat{t}_{\mathrm{ref}}$ be the time of flight from $\mathbf{x}_0^{\mathrm{ref}}$ to the boundary in the absence of noise, i.e.

$$\hat{h}(\phi_1(\mathbf{x}_0^{\mathrm{ref}}, \hat{t}_{\mathrm{ref}})) = 0. \tag{10}$$

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We define Δt_{ref} to be the random variable given by the difference between $\hat{t}_{\mathrm{ref}}$ and the actual time of flight

$$\Delta t_{\mathrm{ref}} = t_{\mathrm{ref}} - \hat{t}_{\mathrm{ref}}. \tag{11}$$

Stochastic Saltation Matrix

In order to deal with stochastically oscillating boundaries we extend the state space, such that the state vector and vector field are given by

$$\tilde{\mathbf{x}} = (\mathbf{x}, t, \quad t_{\mathrm{ref}})^T \quad \text{and} \quad \tilde{\mathbf{f}} = (\mathbf{f}, 1, 0)^T, \tag{12}$$

respectively.

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We calculate the saltation matrix in this extended state space before projecting back. As a result, in the original state space we find that

$$(\mathbf{x}_0; t) \quad (\hat{\mathbf{x}}_0^{\mathrm{ref}}; t) \qquad {}_{\mathbf{x}}(\hat{\mathbf{x}}_0^{\mathrm{ref}}; t)(\mathbf{x}_0 \;\; \hat{\mathbf{x}}_0^{\mathrm{ref}}) + {}_{2;\mathbf{x}}(\hat{\mathbf{x}}_{\mathrm{out}}; t \;\; \hat{t}_{\mathrm{ref}})(\hat{\mathbf{f}}_{\mathrm{in}} \;\; \hat{\mathbf{f}}_{\mathrm{out}}) \;\; t_{\mathrm{ref}}; \tag{13}$$

where

$${}_{\mathbf{x}}(\hat{\mathbf{x}}_0^{\mathrm{ref}}; t) = {}_{2;\mathbf{x}}(\hat{\mathbf{x}}_{\mathrm{out}}; t \;\; \hat{t}_{\mathrm{ref}}) \mathrm{D}_{\mathbf{x}}(\hat{\mathbf{x}}_{\mathrm{in}}) \; {}_{1;\mathbf{x}}(\hat{\mathbf{x}}_{\mathrm{in}}; \hat{t}_{\mathrm{ref}}) \tag{14}$$

and

$$\mathrm{D}_{\mathbf{x}}(\hat{\mathbf{x}}_{\mathrm{in}}) = \mathrm{I} + \frac{(\hat{\mathbf{f}}_{\mathrm{out}} \;\; \hat{\mathbf{f}}_{\mathrm{in}})\hat{h}_{\mathbf{x}}(\hat{\mathbf{x}}_{\mathrm{in}}; \hat{t}_{\mathrm{ref}})}{\hat{h}_{\mathbf{x}}(\hat{\mathbf{x}}_{\mathrm{in}}; \hat{t}_{\mathrm{ref}})\hat{\mathbf{f}}_{\mathrm{in}} + \hat{h}_t(\hat{\mathbf{x}}_{\mathrm{in}}; \hat{t}_{\mathrm{ref}}) \quad V(\hat{t}_{\mathrm{ref}} | P(\hat{t}_{\mathrm{ref}}) = 0)}: \tag{15}$$

In all the above ^ indicates the values associated with the deterministic reference trajectory.

Summary

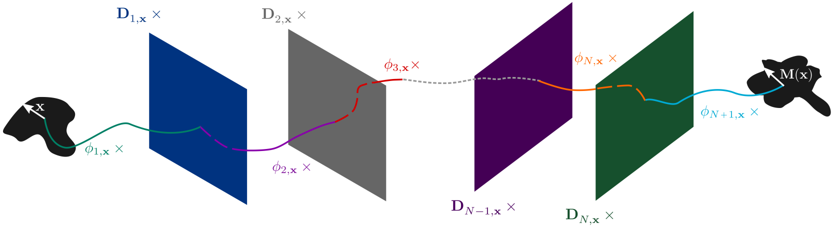
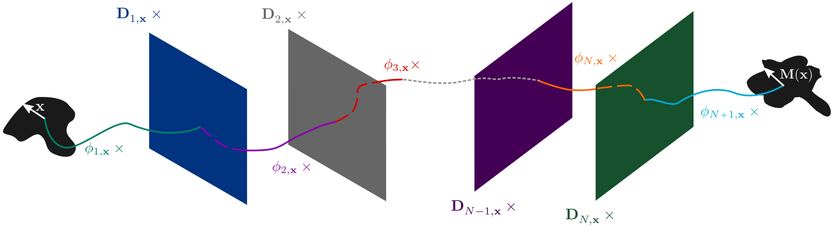


Figure: Linearising Discontinuous Systems

Summary



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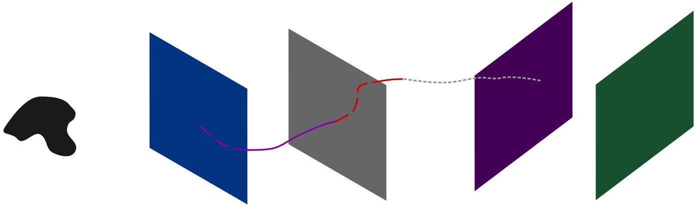

The Chua Circuit

Figure: The Chua Circuit

Figure: The V I characteristic of the Chua Diode.

- Created with the aim of being the simplest autonomous circuit capable of generating chaos [Mat84, Chu92].
- First physical system for which the presenc of chaos was shown experimentally, numerically and mathematically [CKM86].
- Contains four linear elements and one nonlinear resistor known as a Chua diode.
- Easily and cheaply constructed using stadard electronic components [Ken92].

System equations

The dynamics of the Chua circuit can be described by the following nondimensionalised state equations

$$\begin{aligned} \frac{dx}{dt} &= \quad (y \quad x \quad g(x)); \\ \frac{dy}{dt} &= x \quad y + z; \\ \frac{dz}{dt} &= \quad (\; y \; + \; z \,); \end{aligned} \tag{16}$$

where $g(x)$ is the piecewise linear function representing the V-I characteristic of Chua's diode

$$g(x) = \begin{cases} m_1 x + m_1 \quad m_0 & \text{if } x < \quad 1; \\ (m_0 \quad)x & \text{if } |x| \quad 1; \\ m_1 x + m_0 \quad m_1 & \text{if } x > 1. \end{cases} \tag{17}$$

Complicated Dynamics

Figure: A Zoo of Attractors Produced by the Chua Circuit [BP08]

Hidden and Self-Excited Attractors

Hidden attractors: have basins of attraction that do not intersect with small neighborhoods of equilibria.

Self-excited attractors : Can be found by following trajectories from the neighbourhoods of unstable equilibria until the end of a transien process [LK13].

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For a range of parameter values the Chua circuit has a 5-stable regime including 3 hidden periodic attractors [SKLC17].

A Discontinuous Model

Provided the magnitude of is not too large the hidden attractors in the 5-stable regime continue to exist and can be easily found by numerical continuation.

They are destroyed in saddle-bifurcations if the magnitude of grows too large.

Figure: Bifurcation diagram showing the saddle bifurcations of C as the magnitude of grows. Here $= 8{:}4$, $= 12$, $= 0{:}005$, $m_0 = 1{:}2$ and $m_1 = 0{:}145$.

Steady-State Distributions

Figure: Steady state distribution of orbit errors on the discontinuity boundary D for trajectories with initial condition on the periodic orbit C .

Figure: Convergence of ${}_z$ to its steady state value for the distribution shown on the left.

Destroying Periodic Attractors

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A 3-Stable Regime

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For a range of parameter values the Chua circuit has a 3-stable regime including 2 symmetric periodic attractors.

The two periodic attractors merge in a supercritical pitchfork bifurcation if the magnitude grows too large.

Merging Periodic Attractors

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Conclusions

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  - destruction of attractors
  - multi/monostability
  - merging of attractors
  - ickering/switching
- It remains to generalise our method
  - Higher order terms for continuous systems
  - Non-identity boundary mappings
  - Dealing with non-transversal intersections
  - Rugged, stochastic discontinuity surfaces

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