The functional calculus and all that

"Spectral theory" associates sets of complex numbers $\omega(T)$ with linear operators T (or other inscrutables, eg chemical elements): this is as old as the eigenvalues of a matrix and as natural as the range of a function. While the spectrum can only provide a sort of thumbnail sketch of the operator it is capable, through the magic of the Cauchy integral, of putting mathematical flesh on the Heaviside intuition that is the "functional calculus". If "functions of an operator" are a worthwhile project then so are "functions of several operators", which would look for a "spectrum" of several operators at once, finite or infinite sequences or more general systems $(T_t)_{t\in X}$. Such a spectrum would of course be a subset of the systems of complex numbers $\mathbf{C}^X$ indexed by the same set as the operators. It turns out that natural subsets of the spectrum of one operator extend naturally to more general systems - left, right, point, approximate point and defect spectrum etc. These sets turn out for the most part to be compact, in the cartesian product topology, and subject to one half of the spectral mapping theorem for "polynomials". Here the polynomials constitute the free algebra generated by the indexing material, that is the linear space generated by words in the points $t \in X$ of the index set, and a system of such polynomials $p = (p_s)_{s\in Y}$ converts one system of operators into another. Now it usually only needs the remainder theorem to see that $p\omega(T) \subseteq \omega p(T)$, the polynomial image of one of these spectra of a system of operators is a subset of the corresponding spectrum of the image of the operators.

Without commutivity however these spectra may degenerate to the empty set, equality in the spectral mapping theorem may fail, and there is obviously no hope of a functional calculus. In contrast when the operators do mutually commute there is a successful concept of joint spectrum which is guaranteed nonempty, is subject to the full spectral mapping theorem, and supports a functional calculus $f \mapsto f(T)$ valid for functions holomorphic in neighbourhoods of a joint spectrum $\mathrm{Sp}(T)$. Some of the proof is similar to Gelfand theory, calling on complex analysis, in particular the famous Liouville theorem behind the fundamental theorem of algebra. If we could extend the essence of this to non commuting systems of operators, this would be "non commutative spectral theory".

The classical functional calculus, for several variables in a commutative algebra, starts with the spectrum derived from the "maximal ideal space" of the algebra, in the sense of the bounded multiplicative linear functionals

1.1 $$\sigma(A) = HBL(A, \mathbf{C}) = \{\varphi \in A^\dagger : \varphi(a'a) - \varphi(a')\varphi(a) = 0 = 1 - \varphi(1)\} \ .$$

For arbitrary $a \in A^X$ we set

1.2 $$\sigma(a) = \sigma_A(a) = \{\varphi \circ a : \varphi \in \sigma(A)\} \subseteq \mathbf{C}^X \ .$$

If $f \in \mathrm{Poly}_n$ is a polynomial in n complex variables and $a \in A^n$ then it is natural to define $f(a)$ by means of

1.3 $$f = \sum_{j=0}^{n} c_j z^j \Longrightarrow f(a) = \sum_{j=0}^{n} c_j a^j \ .$$

More generally if $f \in \mathrm{Holo}(U)$ is holomorphic on (some open neighbourhood of) a set U for which

1.4 $$\prod_{j=1}^{n} \sigma(a_j) \subseteq U \subseteq \mathbf{C}^n$$

then we can define $f(a)$ by repeating the one variable Cauchy integral formula:

1.5 $$f(a) = (2\pi i)^{-n} \oint_{\sigma(a_n)} \big(\ldots\big(\oint_{\sigma(a_1)} f(z)(z_1 - a_1)^{-1} \ldots (z_n - a_n)^{-1} dz_1\big) \cdots\big) dz_n \ .$$

Sufficient would be for U to be an open complex box containing the product of the spectra: if $\nu \in [0, \infty[^n$ write

1.6 $$\triangle(\nu) = \bigcap_{j=1}^{n} \{|z_j| < \nu_j\} \ .$$

If $U = \triangle(\nu)$ with $\sigma(a_j) \subseteq \triangle(\nu_j)$ for each j then $f \in \mathrm{Holo}(U)$ and hence also $f(a) \in A$ can be given by power series. The trick in the classical commutative algebra case is to extend to "polynomial polyhedra" U, sets of the form

1.7 $$(z, p)^{-1}(\nu, \mu) = \{\lambda \in \mathbf{C}^n : (\lambda, p(\lambda)) \in \triangle(\nu, \mu)\}$$

induced by $(\nu, \mu) \in [0, \infty[^{n+m}$ and $p \in \mathrm{Poly}_n^m$: there is an *Oka extension theorem* which guarantees

1.8 $$f \in \mathrm{Holo}(z, p)^{-1}\triangle(\nu, \mu) \Longrightarrow f = F(z, p) \ \textit{with}\ F \in \mathrm{Holo}\ \triangle(\nu) \ .$$

Evidently we can define $f(a) = F(a, p(a))$, using the Cauchy integrals (1.5) for $F(z, w)$ with $w = p(z)$. In particular we can fit a polynomial polyhedron between the spectrum $\sigma(a)$ and its "polynomially convex hull". Now, for finite systems of commutative algebra elements there is an *Arens-Calderon trick* which associates with $a \in A^n$ another system $b \in A^m$ for which the extended joint spectrum $\sigma(a, b)$ is polynomially convex. Thus finally if $f \in \mathrm{Holo}\ \sigma(a)$ we can write

1.9 $$f(a) = G(a, b, q(a, b)) \ \textit{with}\ G \in \mathrm{Holo}\ \triangle(\nu, \mu, \lambda) \ \textit{and}\ f(z) = G(z, w, q(z, w)) \ .$$

In a sense all this extends to infinite systems $a \in A^X$, if "polynomials" $p : X \to \mathbf{C}$ are interpreted as depending on only finitely many "co-ordinates" z_t $(t \in X)$. Less trivial polynomials live on linear spaces: a homogeneous polynomial of degree n on a linear space E is derived from a multilinear map on the product E^n:

2.1 $$p(x) = p^\wedge(x, x, \ldots, x) \ (x \in E) \ .$$

Lucien Waelbroeck made an extension of the calculus for finite systems to, in effect, systems indexed by the elements of the dual of a Banach space E:

2.2 $$a \in A \otimes_1 E \mapsto a^\vee = \big((I \otimes \psi)(a)\big)_{\psi \in E^\dagger} \ .$$

Here $A \otimes_1 E$ is the "projective" tensor product, completing the finite tensor product with respect to the "greatest crossnorm", and we can define

2.3 $$(S \otimes T)(a) = \lim\{\sum_{j=1}^{\ell} S(\alpha_j)T(\xi_j) : \sum_{j=1}^{\ell} \alpha_j \otimes \xi_j \to a\} \ .$$

More interesting is the case of a commuting system $a \in A^X$ for a non commutative Banach algebra A: now the maximal ideal space is liable to be empty, and we must look elsewhere for a spectrum $\sigma(a)$. One way to proceed is to embed the elements $\{a_t : t \in X\}$ is some natural commutative subalgebra $B \subseteq A$, for example the closed subalgebra generated, or better the *double commutant*: this turns out not to be "canonical" enough, as well as generating an unnecessarily large spectrum. For the particular algebra $A = B(X)$ there turns out to be a natural, somewhat sophisticated, concept due to Joseph L. Taylor: n-tuples $T \in B(X)^n$ generate operator-valued differential forms $\mathbf{T} = \Lambda(T) = \sum_j T_j \otimes dz_j$ acting by exterior multiplication on the space $\mathbf{X} = X \otimes \Lambda(dz) = \Lambda(X, dz)$ of vector valued differential forms. For commuting systems T the differentiator

2.4 $$\mathbf{T}^2 = \mathbf{O}$$

is nilpotent, and may or may not be "exact",

2.5 $$\mathbf{T}^{-1}(0) \subseteq \mathbf{T}(\mathbf{X}) \ ,$$

in which case the system $T \in B(X)^n$ is declared to be *Taylor non singular*. If in particular $\mathbf{T}$ is "splitting exact", in the sense that

2.6 $$\mathbf{VT} + \mathbf{TU} = \mathbf{I} \ ,$$

then the system $T \in B(X)^n$ is declared to be *Taylor invertible*. The Taylor spectrum $\mathrm{Sp}(T)$ is naturally the set of those n-tuples $\lambda \in \mathbf{C}^n$ for which the system $T - \lambda I$ fails to be Taylor non singular, and the sometimes larger Taylor split spectrum $\mathrm{Sp}^{split}(T)$ the set of those n-tuples $\lambda \in \mathbf{C}^n$ for which the system $T - \lambda I$ fails to be Taylor invertible. It is not at all clear what in a more general algebra A should be the "Taylor spectrum" of a commuting n tuple $a \in A^n$, but is entirely clear what should be the "Taylor split spectrum":

2.7 $$\mathrm{Sp}(L_a) = \mathrm{Sp}(R_a) = \mathrm{Sp}_A^{split}(a) \ .$$

The original work of Joseph Taylor established the functional calculus $f(T)$ for functions $f \in \mathrm{Holo}\ \mathrm{Sp}(T)$, although perhaps not in a very constructive form; Kordula and Müller found a more visible version for $f \in \mathrm{Holo}\ \mathrm{Sp}^{split}(T)$. This in particular is available for $f \in \mathrm{Holo}\ \mathrm{Sp}_A^{split}(a)$ for commuting $a \in A^n$.

For infinite systems $a \in A \otimes_1 E$ there is a canonical way of extending a well behaved spectrum ω from finite systems, and if there is a functional calculus for the finite systems that too extends. We begin by looking at abstract "spectral systems", as derived ([Mu] §7) from Vladimir Müller's "regularity" concept. If A is a complex linear algebra then $A^{\mathbf{SET}}$ is the category, whose objects are systems $(a_t)_{t\in X}$ indexed by sets X, with morphisms in the first instance induced by mappings $j : Y \to X$ between sets. We should distinguish two important subcategories: the finite systems

3.1 $$\mathbf{FIN}(A^{\mathbf{SET}}) = \{(a_t)_{t\in X} : \{t \in X : a_t \neq 0\} < \infty\}$$

and the commutative systems

3.2 $$\mathbf{COMM}(A^{\mathbf{SET}}) = \{(a_t)_{t\in X} : \forall\ s,t \in X :\ a_s a_t = a_t a_s\}\ .$$

For arbitrary $a, b \in A^X$ we write

3.3 $$a \cdot b = (a_t b_t)_{t\in X}\ ,$$

and for $a \in \mathbf{FIN}(A^X)$ we write

3.4 $$\sum(a) = \sum_{t\in X} a_t\ .$$

Now a "regularity" in the sense of Müller is a system $\big(\mathbf{R}_X\big)_{X\in\mathbf{SET}}$ of subsets $\mathbf{R}_X \subseteq \mathbf{COMM}(A^X)$ for which

3.5 $$\forall\ (a,b) \in \mathbf{COMM}(A^{X,X})\ :\ a \cdot b \in \mathbf{FIN}(A^X)\ ,\ \sum(b \cdot a) = 1 \Longrightarrow a \in \mathbf{R}_X\ ;$$

3.6 $$\forall\ (a,b) \in \mathbf{COMM}(A^{X,Y})\ :\ a \in \mathbf{R}_X \Longrightarrow (a,b) \in \mathbf{R}_{X,Y}\ ;$$

3.7 $$\forall\ (a,b) \in \mathbf{COMM}(A^{X,Y})\ :\ (a, b - \mathbf{C}^Y) \subseteq \mathbf{R}_{X,Y} \Longrightarrow a \in \mathbf{R}_X\ .$$

There follows an extension of (3.5):

3.8 $$\forall\ (a,b) \in \mathbf{COMM}(A^{X,X})\ :\ a \cdot b \in \mathbf{FIN}(A^X)\ ,\ \sum(b \cdot a) \in \mathbf{R}_{\{1\}} \Longrightarrow a \in \mathbf{R}_X\ .$$

More generally

3.9 $$\forall\ (a,B) \in \mathbf{COMM}(A^{X,Y\times X})\ :\ Ba \in \mathbf{FIN}(A^Y)\ ,\ \sum(Ba) \in \mathbf{R}_Y \Longrightarrow a \in \mathbf{R}_X\ ,$$

and hence

3.10 $$\forall\ a \in \mathbf{COMM}(A^X)\ ,\ \forall j \in Y^X\ ,\ : a \circ j \in \mathbf{R}_Y \Longrightarrow a \in \mathbf{R}_X\ .$$

Towards (3.8) use (3.6) to see that

$$\sum(b \cdot a) \in \mathbf{R}_{\{1\}} \Longrightarrow (\sum(b \cdot a), a) \in \mathbf{R}_{\{1\},X}$$

and claim that

$$0 \neq \lambda \in \mathbf{C} \Longrightarrow (\sum(b \cdot a) - \lambda, a) \in \mathbf{R}_{\{1\},X}\ :$$

this is because

$$(-\lambda^{-1})(\sum(b \cdot a) - \lambda) + \sum(\lambda^{-1} b \cdot a) = 1\ .$$

The same argument extends to (3.9): if $\sum(Ba) \in \mathbf{R}_Y$ and $0 \neq \lambda \in \mathbf{C}^Y$ then there must be $s \in Y$ for which $\lambda_s \neq 0$: now, with Kronecker delta δ,

$$\sum\big(-\lambda_s^{-1}\delta_s \cdot (\sum(Ba) - \lambda)\big) + \sum(\lambda_s^{-1}\delta_s \cdot Ba) = 1\ .$$

Finally (3.10) follows, if we observe

3.11 $$B(s,t) = \delta_{j(s),t} \Longrightarrow Ba = a \circ j\ .$$

Regularities and joint spectra correspond: $\omega = \varpi_{\mathbf{R}}$ and $\mathbf{R} = H_X$ are given by

4.1 $$\omega(a) = \{\lambda \in \mathbf{C}^X : a - \lambda \notin \mathbf{R}_X\} \ ; \ \mathbf{R}_X = \{a \in A^X : 0 \notin \omega(a)\} \ .$$

Now the properties of a regularity induce properties of a joint spectrum: temporarily we write

4.2 $$\sigma^\wedge(a) = \sigma_B(a) \ \textit{with} \ B = \mathrm{cl}\ \mathrm{Poly}_X(a) = \overline{\mathrm{Alg}}(A) \ .$$

A spectral system ω then has the following three properties:

4.3 $$\forall \ a = (a_t)_{t \in X} \in \mathrm{COMM}(A^{\mathbf{SET}}) \ : \ \omega(a) \subseteq \sigma^\wedge(a) \ ;$$

4.4 $$\forall \ a \in \mathrm{COMM}(A^{\mathbf{SET}}) \ , \ \forall \ j \in \mathbf{SET}^{\mathbf{SET}} \ : \ \exists \ a \circ j \Longrightarrow \ \omega(a) \circ j \subseteq \omega(a \circ j) \ ;$$

4.5 $$\forall \ a \in \mathrm{COMM}(A^{\mathbf{SET}}) \ , \ \forall \ j \in \mathbf{SET}^{\mathbf{SET}} \ : \ \exists \ a \circ j \Longrightarrow \omega(a \circ j) \subseteq \omega(a) \circ j \ .$$

For example (4.3) follows from (3.5):

$$a \in \mathrm{COMM}(A^X) \ , \ \lambda \notin \sigma^\wedge(a) \Longrightarrow \exists b \in \overline{\mathrm{Alg}}(a) \ : \textstyle\sum (b \cdot (a - \lambda)) = 1$$

giving $a - \lambda \in \mathbf{R}_X$ and hence $\lambda \notin \omega(a)$.

(4.4) follows from (3.10): if $a \circ j \in \mathbf{R}_Y$ then $(a \circ j, a) \in \mathbf{R}_{Y,X}$, and if $0 \neq \mu \in \mathbf{C}^Y$ then $(a \circ j - \mu, a) \in \mathbf{R}_{Y,X}$:

$$\mu_s \neq 0 \Longrightarrow \textstyle\sum \big(-\mu_s^{-1}\delta_s \cdot (a \circ j - \mu)\big) + \sum \big(\mu_s^{-1}\delta_{j(s)} \cdot a\big) = 1 \ .$$

Conversely for (4.5) argue that if either $a - \lambda \in \mathbf{R}_X$ or $\mu \neq \lambda \circ j$ then $(a \circ j - \mu, a - \lambda) \in \mathbf{R}_{Y,X}$: for

$$\mu_s \neq \lambda_{j(s)} \Longrightarrow \textstyle\sum \delta_s \cdot (a \circ j - \mu) - \sum \delta_{j(s)} \cdot (a - \lambda) = \lambda_{j(s)} - \mu_s \neq 1 \in \mathbf{C} \ .$$

Together (4.4) and (4.5) can be used to extend a spectral system ω from finite systems $a \in A^Y$ to arbitrary systems $a \in A^X$.

There is a trivial regularity; the following are equivalent:

4.6 $$\mathbf{R}_X = \mathbf{COMM}(A^X) \ ;$$

4.7 $$\forall a \in \mathbf{COMM}(A^X) \ , \ \omega(a) \neq \emptyset \ ;$$

4.8 $$\exists a \in \mathbf{COMM}(A^X) \ , \ \omega(a) \neq \emptyset \ ;$$

4.9 $$\{0\} \notin \mathbf{R}_{\{1\}} \ .$$

A more relevant category COMM($A^{\mathbf{SET}}$) has for morphisms those induced not only by by mappings $j : Y \to X$, but also *systems of polynomials* $p : \mathbf{C}^X \to \mathbf{C}^Y$, which in an obvious way induce mappings between commuting systems of algebra elements. Now the spectral mapping theorems (4.4) and (4.5) extend from composition mappings to polynomials: for arbitrary $a \in \mathbf{COMM}(A^X)$ and $p \in \mathrm{Poly}_X^Y$

5.2 $$p\ \omega(a) \subseteq \omega\ p(a)$$

and

5.3 $$\omega\ p(a) \subseteq p\ \omega(a)\ .$$

For $(a, b) \in \mathbf{COMM}(A^{X,Y})$ and $q \in \mathrm{Poly}_{X,Y}^Z$ there is also equality

5.4 $$\omega\ q(a, b) = \bigcup\{\omega(a, s) : s \in \omega(b)\} = \bigcup\{\omega(t, b) : t \in \omega(a)\}\ .$$

From (5.2) we observe a curious phenomenon: if the set X has additional structure, either linear or semigroup, or topological, or differential, and if a system $a \in A^X$ respects this, being either linear, or homomorphic, or continuous, or differentiable, then the same is true of every $\lambda \in \mathbf{C}^X$ in $\omega(a)$. The pattern common to these phenomena seems to be the following: if $\mathbf{F} : \Omega \to \mathbf{SET}$ is a "forgetful functor" and if $a \in A^{\mathbf{SET}}$ then there is implication

5.5 $$a \in A^\Omega\ ,\ \lambda \in \omega(a) \Longrightarrow \lambda \in \mathbf{C}^\Omega\ .$$

For tensor product elements $a \in \mathbf{COMM}(A \otimes_1 E)$ and spectral systems $\omega = \varpi_{\mathbf{R}}$ the "spectrum" $\omega(a^\vee) \subseteq E^{\dagger\dagger}$ of the associated system $a^\vee = \big((I \otimes \psi)(a)\big)_{\psi \in E^\dagger}$ is naturally a subset of the second dual of the Banach space E, but turns out to lie in (the canonical image of) the Banach space E itself, giving us

6.1 $$a \mapsto \omega(a) : \mathbf{COMM}(A \otimes_1 E) \mapsto \mathbf{SUBSET}(E) \ .$$

This is because the mappings

6.2 $$a^\wedge : \varphi \mapsto (\varphi \otimes I)(a) \ (E^\dagger \to E)$$

and

6.3 $$a^\vee : \psi \mapsto (I \otimes \psi)(a) \ (A^\dagger \to A)$$

are (weak*,norm) continuous, not only for $a \in A \otimes_1 E$ but even for $a \in A \otimes_\infty E$. Now we shall say that the spectral system ω "admits a functional calculus" if there is a unique τ_0-continuous homomorphism

6.4 $$\theta_a : \text{Holo}\ \omega\ \overline{Alg}(a) \to A$$

for which

6.5 $$\forall\ \psi \in E^\dagger\ ,\ \theta_a(\psi) = a^\vee(\psi) = (I \otimes \psi)(a)\ ;$$

6.6 $$\forall\ \varphi \in \omega\ \overline{Alg}(a)\ ,\ \forall f \in \text{Holo}\ \omega(a)\ ,\ \varphi(\theta_a(f)) = f(a^\wedge(\varphi)) = f((\varphi \otimes I)(a))\ .$$

"Non commutative topology" resides in the observation that a commutative C* algebra A is to all intents and purposes $C_0(\Omega)$ the continuous functions vanishing at infinity on a compact Hausdorff topological space Ω, with the space determined uniquely by the algebra. Thus any topological property of such a space corresponds to an algebraic property: for example the space is actually compact iff the algebra has an identity, and idempotents in the algebra correspond to open-and-compact subsets of the space. If we now pass to non commutative C* algebras, for which no such spaces exist, and if we can convincingly identify "the same" algebraic properties in the non commutative environment, then it is tempting to carry over the topological language, and thus to build up a shadowy picture of a "virtual" topological space. This language seems to be particularly appropriate to the quantum theory, where the Euclidean certainties of Newtonian space and time have been replaced by something much less intuitive. In any event "non commutative topology" is sort of busy.

In the literature on non commutative topology there seem to be three important notions: operator spaces, Hilbert modules, and Hopf algebras. As a taster for "non commutative spectral theory", one might look at commutative spectral theory in each case. An "operator space" [ER] can be realised as a closed subspace of the bounded operators on a Hilbert space X, not necessarily closed under either multiplication or adjoints. $M \subseteq B(X) = BL(X, X)$ comes with a retinue of "matricial extensions" $M^{n\times n} \subseteq B(X^{n\times n})$, and bounded linear operators $T \in BL(M, N)$ have matricial extensions $T^{n\times n} \in BL(M^{n\times n}, N^{n\times n})$. While each such extension is also bounded, the bound is liable to increase with n, and a "completely bounded" operator T is one for which this increase is bounded. It turns out that the completely bounded operators on an operator space M form a Banach algebra $CB(M)$ in their own right, and that therefore classical spectral theory is available. While the "completely bounded spectrum" of $T \in CB(M)$ is potentially larger than its spectrum in $B(M)$, we can see that its isolated points at least must be in the usual spectrum.

A "Hilbert module" M, over a specific C* algebra A, is [L] either a left or a right Banach A module, so that in the usual Banach space theory the "scalars" have been replaced by Banach slgebra elements, and the obvious norm inequality $\|am\| \leq \|a\|\,\|m\|$ imposed. When A is a C* algebra then for a Hilbert module we ask that the norm be derived from the analogue of an inner product, $\langle \cdot ; \cdot \rangle : M \times M \to A$, with appropriate positivity properties. At first sight the natural mappings $T : M \to N$ between Hilbert modules would be the bounded A-linear mappings, for which $T(am) = aT(m)$, and one would then look for an "adjoint", $T^* : N \to M$ for which $\langle Tm; n \rangle = \langle m; T^* n \rangle$. It turns out that this need not always exist, and that when it does it guarantees A-linearity: thus between Hilbert modules it becomes natural to consider such "adjointable" linear mappings. On a specific Hilbert module the adjointable linear mappings form another C* algebra: thus it seems that the commutative spectral theory here is nothing new.

Finally a "Hopf algebra" A comes not only with a multiplication, which is a linear mapping $\cdot : A \otimes A \to A$ from the tensor square back into the algebra, but also a "co multiplication" $\triangle : A \to A \otimes A$ going the other way. For Banach algebras, the multiplication will be bounded on the "projective" tensor product $A \otimes_1 A$, and it would seem reasonable to expect the co multiplication to be bounded into the dual "inductive" product $A \otimes_\infty A$. Properties of a "co multiplication" are derived from expressing the associative law of multiplication, the concept of an identity, invertibility and commutivity etc, in terms of the tensor product diagram, and then "reversing the arrows". Thus one might ask whether there is a "co spectral theory", collecting complex numbers $\lambda \in \mathbf{C}$ for which $a - \lambda$ fails to be "co invertible" and attacking such a thing with "co polynomials". In a Banach Hopf algebra one would be tempted to set such a co spectrum beside the usual spectrum and compare notes; alternatively the co spectrum of $a \in A$ might well be related to the usual spectrum of the image $\triangle(a) \in A \otimes A$.

The discussion of these variations on the usual spectrum of an operator would constitute a natural "five finger exercise" to help us to read our way into non commutative topology, but probably only a side show on the way to a proper understanding of "non commutative spectral theory". Other straws in the wind would be the work of Vladimir Kisil, who [K1], [K2] replaces the commutative functional calculus based on a homomorphism out of a function algebra by an intertwining of group representations, and the "quasideterminants" of Gelfand, Gelfand, Retakh and Wilson, which [GGRW] seem to be some kind of non commutative analogue of the cofactor matrix [HH], [AM]. Our own intuition is to go back to the concept of a "system of elements" $a = (a_t)_{t \in X}$ in a Banach algebra A, with a possibly uncountable index set Ω. Such Ω can now support algebraic or topological structure, and the system a can therefore be either a homomorphism or a continuous mapping. The small miracle is then [H1],[H2] that the one way spectral mapping theorem imposes on systems $\lambda = (\lambda_t)_{t \in X}$ belonging to the spectrum of a all the same such homomorphic or continuity properties (5.5). In particular a Banach algebra A can be thought of as the system $(a)_{a \in A}$ indexed by itself, and hence everything in the spectrum recognised as multiplicative linear functionals. This means that the spectral mapping theorem for commuting systems of elements incorporates the classical Gelfand theory. Our new idea would be to try and visualise systems $a = (a_t)_{t \in X}$ in which the index set X was only "virtual", or "quantized", and try to extend spectral mapping theorems and functional calculi to this context.

One extension of the idea of a system $a = (a_t)_{t \in X}$ of Banach algebra elements would be to the idea of a functor, $a : \Omega \to A$, from a category Ω into a "Banach linear category" A. Intuitively a *Banach linear category* is just a very big Banach algebra with a partially defined multiplication: two examples which spring to mind would be the rectangular matrices over a specific Banach algebra, and the category of all bounded linear operators between Banach spaces. It would be nice if completely bounded mappings between operator spaces, and adjointable mappings between C* modules, would fall under this umbrella. There is already in the literature [GLR],[Mi] something called a "C* category". We would now look for "polynomials in Ω variables", and what we would mean by a "spectrum", sending systems $a = (a_t)_{t \in \Omega}$ into an analogue of the category of all subsets of a set Ω, and seek to establish both forward and backward spectral mapping theorems. Related to polynomials would be "holomorphic functions", with an aspiration to a functional calculus.

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Suppose A is a complex topological algebra, with identity 1, locally convex, jointly continuous multiplication, each compact subset contained in some convex compact, and for open $U \subseteq \mathbf{C}^n$ look at holomorphic $f : U \to A$, written $f \in \mathrm{Hol}(U, A)$. If A is commutative, declare $U \subseteq \mathbf{C}^n$ to be an *elementary resolvent set* for $a \in A^n$ if there exist $g \in \mathrm{Hol}(U, A)^n$ for which

*A*1 $$\sum_{j=1}^{n} g_j(z)(z_j - a_j) = 1 \text{ on } U \ .$$

Write $\rho_A(a)$ for the *resolvent set* of $a \in A^n$, defined as the union of all the elementary resolvent sets, whose complement $\sigma_A^{\sim}(a)$ is the *analytic joint spectrum* of $a \in A^n$, including of course the joint spectrum

*A*2 $$\sigma_A(a) = \{\lambda \in \mathbf{C}^n : 1 \notin \sum_{j=1}^{n} A(\lambda_j - a_j)\} \ .$$

For a Banach algebra they coincide: for if $\sum_j b_j(a_j - \lambda_j) = 1$ then $\sum_j b_j(a_j - \mu_j) = f(\mu)^{-1} \in A^{-1}$ for μ in a neighbourhood of λ: now take $g_j(z) = -b_j f(z)$ on this neighbourhood. When (A1) holds we describe g as an *elementary resolvent system* for $a \in A^n$. If $(g_{(k)}, U_k)_{k=1,2,\ldots,N}$ is a sequence of elementary resolvent systems for $a \in A^n$ and if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is an ordered subset of the k_j we shall write

*A*3 $$U_\alpha = \bigcap_{i=1}^{m} U_i \ ; \ Q_\alpha = \det(g_{\alpha_i j}) : U_\alpha \to A \ .$$

Proposition *If $\beta = (\beta_0, \beta_1, \ldots, \beta_m)$ and if β'_j is obtained by deleting β_j from β then*

*A*4 $$Q_{\beta_0} - Q_{\beta_1} + \ldots + (-1)^n Q_{\beta_n} = 0 \ .$$

Proof. By (A1) the matrix
$$\begin{pmatrix} 1 & g_{\beta_0 1} & g_{\beta_0 2} & \cdots & g_{\beta_0 m} \\ 1 & g_{\beta_1 1} & g_{\beta_1 2} & \cdots & g_{\beta_1 m} \\ 1 & \cdots & \cdots & \cdots & \cdots \\ 1 & g_{\beta_m 1} & g_{\beta_m 2} & \cdots & g_{\beta_m m} \end{pmatrix}$$
has determinant zero •

If $\mathbf{K}$ is an abstract complex with vertices $\{1, 2, \ldots, N\}$ and simplices $\beta = (\beta_0, \beta_1, \ldots, \beta_m)$, associate with $a \in A^n$ and n tuples of m simplices an "a-chain"

*A*5 $$\gamma = \sum_{j=1}^{n} a_j \beta_j \ ,$$

identifying simplices which are even permutations of one another etc, and write

*A*6 $$Q_\gamma = \sum_{j=1}^{n} a_j Q_{\beta_j} \ .$$

With
$$\partial(\beta_0, \beta_1, \ldots, \beta_m) = (\beta_1, \ldots, \beta_m) - (\beta_0, \beta_2, \ldots, \beta_m) + \ldots + (-1)^m(\beta_0, \beta_1, \ldots, \beta_{m-1})$$
we have

*A*7 $$Q_{\partial\gamma} = 0 \text{ on } U_\gamma \ .$$

Now suppose $f \in C(V_1 \cup V_2 \cup \ldots V_N, A)$ and define differential forms

*A*8 $$Q_\alpha f dz = Q_\alpha f dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n \text{ on } U_\alpha \ :$$

*A*9 $$Q_{\partial\gamma} f dz = 0 \textit{ on } U_\gamma \ .$$

Now if $\mathbf{K} \subseteq \mathbf{C}^n$ is a polyhedral complex, with subcomplexes $\mathbf{K}_j \subseteq U_j$ $(j = 1, 2, \ldots, N)$, put

$$\omega_\alpha(k) = \int_k Q_\alpha f dz$$

giving an n cochain in $\mathbf{K}_\alpha$, which since Q_α is holomorphic, is a cocycle:

*A*10 $$\omega_\alpha(\partial h) = 0 \textit{ for each } n+1 \textit{ chain } h \textit{ in } \mathbf{K}_\alpha \ .$$

This is the A valued analogue of the Cauchy-Green-Stokes theorem.
Theorem *If $\partial g \subseteq \mathbf{K}_1 \cup \mathbf{K}_2 \cup \ldots \cup \mathbf{K}_N$ then, with δ the coboundary operator,*

$$\partial g_\alpha = g_{\delta\alpha} \ .$$

Lemma *If $a \in A^n$, if $K \subseteq \mathbf{C}^n$ is compact and if $V \in \mathrm{Nbd}(K)$ with*

$$V \setminus K \subseteq \rho_A(a)$$

then there is continuous linear $J_K : \mathrm{Hol}(V, A) \to A$ for which, for each $f \in \mathrm{Hol}(V, A)$,

$$J_K(f) = \frac{1}{(2\pi i)^n} \frac{1}{n!} \sum_\alpha \int_{g_\alpha} Q_\alpha f dz$$