RESEARCH STATEMENT OF ALEXANDER D. RAHM

1. Research axis *Torsion in the homology of discrete groups*

For my projects studying torsion in the homology of discrete groups, I have several collaborators,

- Ethan Berkove (Lafayette College, Pennsylvania),
- Tuan Anh Bui (Ho Chi Minh City University of Science, Vietnam),
- Grant Lakeland (Eastern Illinois University),
- Matthias Wendt (Universität Duisburg-Essen),

who have joined me because of the novel technique of *torsion subcomplex reduction*, which I have developed. It is a technique for the study of discrete groups, which I have first put to work in [8] for a specific class of arithmetic groups: the Bianchi groups, for which my method has yielded all of the homology above the virtual cohomological dimension. Some elements of this technique had already been used by Soulé for a modular group [19]; and were used by Mislin and Henn as a set of ad hoc tricks. After rediscovering these ad hoc tricks, I had success in putting them into a general framework [11]. The advantage of using a systematic technique rather than a set of ad-hoc tricks, is that instead of merely helping isolated example calculations, it becomes possible to find general formulae, as I did for instance for all of the Bianchi groups for all of the homology above their virtual cohomological dimension.

It is convenient to give some examples of where the technique of torsion subcomplex reduction has already produced good results:

- The Bianchi groups,
- The Coxeter groups,
- The PSL_2 groups over arbitrary number rings.

The Bianchi groups. In the case of the Bianchi groups (the PSL_2 groups over rings of imaginary quadratic integers), the torsion subcomplex reduction technique has permitted me to find a description of the cohomology ring of these groups in terms of elementary number-theoretic quantities [11]. The decisive step has been to extract, using torsion subcomplex reduction, the essential information about the geometric models, and then to detach this information completely from the model. I was hence able to show that this information is contained in *conjugacy classes graphs*, which I construct for this purpose for an arbitrary group from its system of conjugacy classes of finite subgroups.

The Coxeter groups. Recall that the Coxeter groups are generated by reflections; and their homology consists uniquely of torsion. Torsion subcomplex reduction hence allows to obtain all of the homology of all of the tetrahedral Coxeter groups at all odd prime numbers, both in a general formula and in terms of explicit tables [11].

The PSL_2 groups over arbitrary number rings. In joint work, Matthias Wendt and I have established formulae for the Farrell-Tate cohomology with odd torsion coefficients for all groups $\mathrm{PSL}_2(A)$, where A is a ring of S-integers in an arbitrary number field [16]. Wendt has furthermore extended this to the cases where A is the ring of functions on a smooth affine curve over an algebraically closed field. These two results together have allowed Wendt to find a refined version of the Quillen conjecture, which keeps track of all the types of known counter-examples to the original Quillen conjecture [17]. So if

there does not exist any counter-example of completely new type to the original Quillen conjecture, then the Quillen–Wendt conjecture must be true.

Adaptation of the technique for SL_2. With Ethan Berkove, I have extended my technique of torsion subcomplex reduction, which originally had been designed for groups with trivial centre (e.g., PSL_2), to be able to treat now also groups with non-trivial centre (e.g., SL_2). This way, we have determined the 2-torsion in the cohomology of the SL_2 groups over the imaginary quadratic number rings [2].

Congruence subgroups. With Ethan Berkove and Grant Lakeland (Eastern Illinois University), I have extended these investigations from the SL_2 groups to their congruence subgroups. This latter collaborator has provided us a sample of computations of fundamental domains for congruence subgroups; and we have proved several theorems which hold for all congruence subgroups [1].

Extension of the technique for PSL_3. The next step of my project with Matthias Wendt is to get torsion subcomplex reduction to work for PSL_3 over the quadratic rings of integers. In order to resolve the technical difficulties specific to PSL_3, the Institut des Hautes Etudes Scientifiques (IHES, France) did invite me for a one-month research stay in 2015, where I had the opportunity to ask questions to Christophe Soulé, who is an expert for the special case $(\mathrm{P})\mathrm{SL}_3(\mathbb{Z})$ [19].

Amalgams. With Graham Ellis (NUI Galway) and Tuan Anh Bui, I import cell complexes into HAP, the *Homological Algebra Programming* package of GAP. An example of the results obtained jointly with Ellis is included in [8]. With Tuan Anh Bui, I have used these machine computations in order to verify the cohomology ring of a congruence group which can be amalgamed into two copies of the group $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}\,])$, in a way to obtain $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}\,][\frac{1}{2}])$. Subsequently, I have used the methods which I have developed jointly with Ethan Berkove, mentioned above, for successfully determining the modulo 2 cohomology ring of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}\,][\frac{1}{2}])$, which plays an important rôle in a project of Hans-Werner Henn (Université de Strasbourg), pursued by Tuan Anh Bui.

Application to equivariant K-homology. In a recent paper [13], I have, for the Bianchi groups, transplanted the torsion subcomplex reduction technique from group homology to Bredon homology with coefficients in the complex representation rings, and with respect to the family of finite subgroups. This has lead me to formulae for this Bredon homology, and by the Atiyah–Hirzebruch spectral sequence, to formulae for equivariant K-homology of the Bianchi groups acting on their classifying space for proper actions. As the Baum–Connes assembly map from the equivariant K-homology to the K-theory of the reduced C^*-algebras of the Bianchi groups is an isomorphism, I obtain the isomorphism type of the latter operator K-theory, which would be extremely hard to compute directly from its definition.

Investigating beyond the range of arithmetic groups, my collaborators Ruben Sanchez-Garcia (University of Southampton), Jean-Francois Lafont (Ohio State University) Ivonne Ortiz (Miami University, Oxford, Ohio) and I have established formulas for the integral Bredon homology and equivariant K-homology of all compact 3-dimensional hyperbolic reflection groups [4]. In that paper, I also have proven a novel criterion for torsion-freeness of equivariant K-homology, and consequently for K-theory of reduced C^*-algebras.

2. Construction of explicit elements in algebraic K-theory

With Rob de Jeu (VU University Amsterdam), I am searching for explicit non-trivial elements in $\mathrm{H}_3(\mathrm{GL}_2(\mathbb{Q}(\sqrt{-m})))$, which can be lifted to the algebraic K-theory of rings of imaginary quadratic integers. Rob de Jeu has found a method for expressing these elements in terms of geometric images coming from the Bianchi groups. *I have determined the latter images, leaning on my answer to a question of Jean-Pierre Serre, which had been open for 40 years* [7]. The explicit description of algebraic K-theory, which we obtain this way, is described in our joint manuscript [3].

The MFO did invite us with two supplementary collaborators, Herbert Gangl (Durham University) and Dan Yasaki (University of North Carolina at Greensboro), for a *Research in Pairs* project in Oberwolfach, permitting us to work on the extension of our method towards the algebraic K-theory of rings of integers in arbitrary number fields.

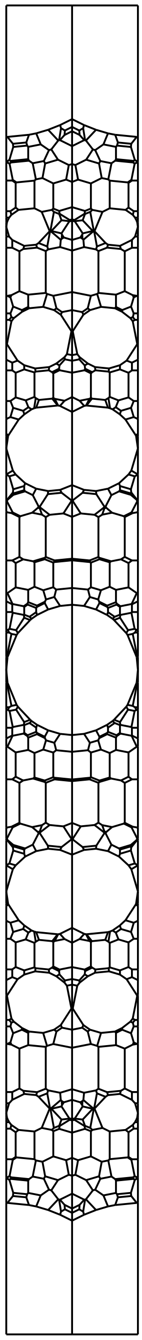

Figure 1. Fundamental domain for $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-102}])$, computed with [10].

3. Bianchi modular forms

With Mehmet Haluk Şengün (University of Sheffield), I search for automorphic representations of the Bianchi groups which cannot be obtained as lifts via the Langlands base change procedure, see Figure 1 and our article [14]. I have obtained a grant of 900,000 processor hours at the ICHEC (Irish Centre for High-End Computing), with which we have extended this research to the automorphic representations of congruence subgroups. With Panagiotis Tsaknias (Université du Luxembourg), I have analysed the computed dimensions of the cuspidal cohomology spaces [15].

4. Chen/Ruan orbifold cohomology

With Fabio Perroni (University of Trieste), I have recently proved Ruan's cohomological crepant resolution conjecture for all complexified Bianchi orbifolds. This means that my computations of the Chen–Ruan cohomology of the complexified Bianchi orbifolds yield the cohomology ring structure on a crepant resolution of the respective orbifold with quantum corrections. They are being integrated into a forthcoming paper with Perroni [6].

5. Modulo prime power modular forms

With Gabor Wiese (University of Luxembourg), I am going to evaluate a database on modulo prime power modular forms, which my collaborator Panagiotis Tsaknias has built up during several years. This is going to produce precise, quantitative conjectures about modulo prime power modular forms, on which I will work with the pertinent knowledge that I have gained in Luxembourg's number theory workgroup.

6. Persistent homology in soft matter physics simulations

Tanja Schilling (University of Freiburg)'s research group develops computer simulation methods to study complex fluids. Using my background in persistent homology which I have acquired in Graham Ellis' research group in Galway (I already have refereed a persistent homology paper), I am going to build a persistent homology computation into one of these simulations, which will allow observations on how the persistent homology develops during the simulated process.

References

[1] Ethan Berkove, Grant Lakeland, and Alexander D. Rahm, *The mod 2 cohomology rings of congruence subgroups in the Bianchi groups*, arXiv: 1707.06078 [math.KT] (2017). With an appendix by Tuan Anh Bui and Sebastian Schönnenbeck.

[2] Ethan Berkove and Alexander D. Rahm, *The mod 2 cohomology rings of* SL_2 *of the imaginary quadratic integers*, J. Pure Appl. Algebra **220** (2016), no. 3, 944–975, DOI 10.1016/j.jpaa.2015.08.002. With an appendix by Aurel Page. MR3414403

[3] Rob de Jeu and Alexander D. Rahm, *The image of the Borel-Serre bordification in algebraic K-theory*, preprint, `http://hal.archives-ouvertes.fr/hal-00975454`.

[4] J.-F. Lafont, I. J. Ortiz, A. Rahm, and R. J. Sánchez-García, *Equivariant K-homology for hyperbolic reflection groups*, arXiv: 1707.05133 [math.KT] (2017).

[5] Guido Mislin and Alain Valette, *Proper group actions and the Baum-Connes conjecture*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2003. MR2027168 (2005d:19007), Zbl 1028.46001

[6] Fabio Perroni and Alexander D. Rahm, *On the Cohomological Crepant Resolution Conjecture for the complexified Bianchi orbifolds*, preprint, `http://hal.archives-ouvertes.fr/hal-00627034/`.

[7] Alexander D. Rahm, *On a question of Serre*, Comptes Rendus Mathématique de l' Académie des Sciences - Paris **350** (2012), 741–744, presented by Jean-Pierre Serre.

[8] ———, *The homological torsion of* PSL_2 *of the imaginary quadratic integers*, Trans. Amer. Math. Soc. **365** (2013), no. 3, 1603–1635, DOI 10.1090/S0002-9947-2012-05690-X. MR3003276

[9] ———, *Homology and K-theory of the Bianchi groups (Homologie et K-théorie des groupes de Bianchi)*, Comptes Rendus Mathématique de l' Académie des Sciences - Paris **349** (2011), no. 11-12, 615–619.

[10] ———, *Bianchi.gp*, Open source program (GNU general public license), validated by the CNRS: `http://www.projet-plume.org/fiche/bianchigp` subject to the Certificat de Compétences en Calcul Intensif (C3I) and part of the GP scripts library of Pari/GP Development Center, 2010.

[11] ———, *Accessing the cohomology of discrete groups above their virtual cohomological dimension*, J. Algebra **404** (2014), 152–175. MR3177890

[12] ———, *Higher torsion in the Abelianization of the full Bianchi groups*, LMS J. Comput. Math. **16** (2013), 344–365. MR3109616

[13] ———, *On the equivariant K-homology of PSL_2 of the imaginary quadratic integers*, Annales de l'Institut Fourier **66** (2016), no. 4, 1667–1689.

[14] Alexander D. Rahm and Mehmet Haluk Şengün, *On level one cuspidal Bianchi modular forms*, LMS J. Comput. Math. **16** (2013), 187–199, DOI 10.1112/S1461157013000053. MR3091734

[15] Alexander D. Rahm and Panagiotis Tsaknias, *Genuine Bianchi modular forms of higher level, at varying weight and discriminant*, arXiv: 1703.07663 [math.NT] (2017).

[16] Alexander D. Rahm and Matthias Wendt, *On Farrell-Tate cohomology of SL_2 over S-integers*, `https://hal.archives-ouvertes.fr/hal-01081081`, preprint.

[17] ———, *A refinement of a conjecture of Quillen*, Comptes Rendus Mathématique de l'Académie des Sciences **353** (2015), no. 9, 779–784, DOI http://dx.doi.org/10.1016/j.crma.2015.03.022.

[18] Jean-Pierre Serre, *Le problème des groupes de congruence pour SL_2*, Ann. of Math. (2) **92** (1970), 489–527. MR0272790 (**42** #7671), Zbl 0239.20063

[19] Christophe Soulé, *The cohomology of* $\mathrm{SL}_3(\mathbf{Z})$, Topology **17** (1978), no. 1, 1–22.

Mathematics Research Unit, Université du Luxembourg
Alexander.Rahm@uni.lu `http://math.uni.lu/~rahm/`